Journal of Statistical Physics, Vol. 102, Nos. 3/4, 2001

Non-Abelian Quantum Hall States—Exclusion Statistics, *K*-Matrices, and Duality[1]

Eddy Ardonne,[2] Peter Bouwknegt,[3] and Kareljan Schoutens[4]

Received April 11, 2000; final June 30, 2000

We study excitations in edge theories for non-abelian quantum Hall states, focussing on the spin polarized states proposed by Read and Rezayi and on the spin singlet states proposed by two of the authors. By studying the exclusion statistics properties of edge-electrons and edge-quasiholes, we arrive at a novel *K*-matrix structure. Interestingly, the duality between the electron and quasihole sectors links the pseudoparticles that are characteristic for non-abelian statistics with composite particles that are associated to the "pairing physics" of the non-abelian quantum Hall states.

KEY WORDS: Fractional quantum Hall effect; exclusion statistics; quasiparticle; *K*-matrix; duality; conformal field theory; character.

1. INTRODUCTION

The fractional quantum Hall effect has led to the identification of new states of matter, which can be characterized as incompressible quantum fluids with off-diagonal long-range order ("topological order"). After the initial discovery of the "principal Laughlin series" of quantum Hall fluids at filling factor $\nu = 1/m$, a large class of so-called abelian quantum Hall fluids has been identified, accounting for the rich spectrum of fractional quantum Hall plateaus that have been observed in the lowest Landau level.

---

[1] Dedicated to Rodney J. Baxter on his 60th birthday.

[2] Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands; e-mail: ardonne@wins.uva.nl

[3] Department of Physics and Mathematical Physics, University of Adelaide, Adelaide SA 5005, Australia; e-mail: pbouwkne@physics.adelaide.edu.au

[4] Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands; e-mail: kjs@wins.uva.nl

0022-4715/01/0200-0421$19.50/0

The observation of a quantum Hall plateau at filling factor $\nu = 5/2$ (see ref. 1 for a recent experiment) has made it clear that the traditional set of abelian quantum Hall states (which all share the property of having an odd-denominator filling factor) will not suffice for explaining the phenomena observed in the second Landau level. Prompted by this development, new categories of incompressible quantum fluids have been proposed. Among them are various "paired" or "clustered" states, such as the Pfaffian states first proposed by Moore and Read.[2] The quasiparticles over these states satisfy what is called non-abelian braid statistics, and by abuse of language one speaks of "non-abelian quantum Hall states."

The characteristic order of the abelian quantum Hall fluids should be viewed as "topological" and it can be characterized by a collection of integer numbers, which together constitute a so-called K-matrix. Many of the low energy characteristics of the Hall fluid are encoded in this K-matrix and the quantum numbers of the elementary electron-type excitations. They include the filling factor ν and the spin Hall conductance σ. In addition the (fractional) quantum numbers of the various quasihole type excitations are determined by using (the inverse of) the K-matrix.

A systematic framework for the physical implications of the topological order embodied in the K-matrix is provided by effective Chern–Simons and conformal field theories for bulk and edge excitations, respectively. In a systematic treatment of the low energy dynamics, these theories arise as special limits of a unifying field theory for the low energy behaviour of quantum Hall systems.[3]

It is well-known that bulk-excitations over fractional quantum Hall fluids satisfy fractional (anyonic) braid statistics. Closely related to this are the fractional exclusion statistics of both bulk[4] and edge excitations.[5, 6] It has been observed[7, 6] that for abelian quantum Hall fluids, the (edge) statistics matrix $\mathbf{K}$ (in the sense of Haldane's definition of exclusion statistics[8]) is closely related to the K-matrix.

The main purpose of the present paper is to present a K-matrix structure associated to specific series of non-abelian quantum Hall states. To this end, we study the exclusion statistics of edge excitations over these quantum Hall states, and identify from that analysis statistics matrices $\mathbf{K}$. We shall then argue that these same matrices can be viewed as K-matrices for these non-abelian quantum Hall states.

Our analysis here builds on earlier results published in refs. 9–13. In ref. 14 we presented our present results in brief form, and elaborated on the physical meaning of the newly obtained K-matrices. An important claim, which we make on the basis of the K-matrix structure here presented, is that there is a direct logical link between the non-abelian statistics of the fundamental quasiparticles and the pairing or clustering of the fundamental electrons that constitute a non-abelian quantum Hall state. This link takes the form of a duality transformation, connecting the pseudoparticles that are responsible for the non-abelian statistics with the composite particles originating from the pairing or clustering of fundamental electrons.

This paper is organized as follows. In Section 2 we briefly review the *K*-matrix theory for abelian quantum Hall states, and make a generalization in order to be able to treat spin singlet states. We continue in Section 3 by making the link to abelian exclusion statistics. We argue that the statistics matrix is (basically) given by the *K*-matrix. Also, we introduce an important notion of duality. In Section 4 we generalize this concept to the non-abelian case, where composites and pseudoparticles play a vital role. It is argued that the well known formulas for the physical quantities such as the filling factor, derived from the abelian *K*-matrix structure, still hold for the non-abelian *K*-matrices describing various clustered non-abelian quantum Hall states. Section 5 deals with the relation between the universal chiral partition function (UCPF) and exclusion statistics. In Sections 6 and 7 the *K*-matrices for two classes of non-abelian clustered states are identified. Section 8 is reserved for discussions, while some of the more mathematical results, for instance on character formulas, are discussed in the appendices.

2. *K*-MATRICES FOR ABELIAN QUANTUM HALL STATES

In this section, we briefly review the *K*-matrix structure for abelian quantum Hall states. We do not derive, but merely state the results we need in this paper. For a more detailed review, see for instance ref. 15.

The information needed to describe an abelian quantum Hall state can be encoded in the following way, henceforth referred to as the fqH data. The four important "objects" which will do the job are the *K*-matrix (which will also play the role as statistics matrix), the so called charge and spin vectors, $\mathbf{t}$ and $\mathbf{s}$, respectively, and the angular momentum vector $\mathbf{j}$. A few remarks with respect to the notation of spin vectors need to be made at this point. In refs. 16 and 17, the concept of a "spin vector" was introduced. This "spin vector" is in fact related to the angular momentum of the electrons on (for instance) the sphere and is needed to calculate the so-called shift. In our case we need to distinguish between this angular momentum vector and the vector containing the real spin of the particles. Therefore, we have denoted the angular momentum vector by $\mathbf{j}$, and the vector containing the spin quantum numbers by $\mathbf{s}$.

In order to have the possibility to connect the *K*-matrix with the statistics matrix (as we will do in the following sections), we will distinguish between the *K*-matrix for the "electron part" and the "quasihole part" of the theory. These will be denoted by $\mathbf{K}_e$ and $\mathbf{K}_\phi$, respectively. The corresponding charge, spin and angular momentum vectors are $\mathbf{t}_e$, $\mathbf{t}_\phi$, $\mathbf{s}_e$, $\mathbf{s}_\phi$, $\mathbf{j}_e$, and $\mathbf{j}_\phi$ in an obvious notation. In all the cases we considered, it is possible to choose a basis in which the K-matrices are just each others inverse, $\mathbf{K}_e = \mathbf{K}_\phi^{-1}$.

As stated above, the K-matrices will play several roles in the theory. First of all, they couple the different Chern–Simons gauge fields which play a central role in a Lagrangian description of the quantum Hall states. In the abelian case, the Chern–Simons part of the Lagrangian for a system on a surface of genus g reads as follows

$$\mathscr{L}_{\text{CS}} = \frac{1}{4\pi} \varepsilon^{\mu\nu\lambda} (\mathbf{K}_e^{ij} a_\mu^i \, \partial_\nu a_\lambda^j + 2\mathbf{t}_e^i A_\mu \, \partial_\nu a_\lambda^i + 2\mathbf{j}_e^i \omega_\mu \, \partial_\nu a_\lambda^i + 2\mathbf{s}_e^i \beta_\mu \, \partial_\nu a_\lambda^i) \tag{2.1}$$

where the fields a are the Chern–Simons gauge fields. The Greek indices run over $\{0, 1, 2\}$, and the Roman indices over the number of channels. The first three terms in Eq. (2.1) are rather standard and described in, for instance, refs. 15–17. The first term is the famous Chern–Simons term, the other three describe the couplings to various fields. The gauge field A_μ describes the electromagnetic field and ω_μ is the "spin connection" which gives rise to the curvature of the space on which the system is defined. To explain the last term, we briefly discuss the concept of the spin Hall conductance and the related spin filling factor σ (see ref. 18 and references therein).

In general, one would define the spin conductance in the same way as the charge conductance, namely as a response to a certain field. In the case of a quantum Hall system, the role of the electric field is taken over by a gradient in the Zeeman energy. The gauge field describing this is denoted by β_μ in Eq. (2.1). The spin Hall conductance is then related to the "spin-current" induced perpendicular to the direction of the gradient of the Zeeman energy.

Let us now briefly recall the results obtained from this formulation for the filling factors and the shift corresponding to a surface of genus g. The filling factors can be calculated by means of simple inner products[5]

$$\begin{aligned} \nu &= \mathbf{t}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{t}_e = \mathbf{t}_\phi \cdot \mathbf{K}_\phi^{-1} \cdot \mathbf{t}_\phi \\ \sigma &= \mathbf{s}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{s}_e = \mathbf{s}_\phi \cdot \mathbf{K}_\phi^{-1} \cdot \mathbf{s}_\phi \end{aligned} \tag{2.2}$$

[5] Throughout this paper the transpose in equations like (2.2) is implicitly understood in order to simplify the notation.

The relation between the charge (and spin) vectors of the electron and quasihole parts are given by

$$\mathbf{t}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{t}_e, \qquad \mathbf{s}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{s}_e \tag{2.3}$$

The last important property we will discuss is the so called "shift" in the flux on surfaces of general genus g. The relation between the number of electrons N_e and the corresponding number of flux quanta N_Φ is given by

$$N_\Phi = \frac{1}{\nu} N_e - \mathcal{S} \tag{2.4}$$

where the shift $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{2(1-g)}{\nu} (\mathbf{t}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{j}_e) \tag{2.5}$$

Although $\mathbf{j}_e$ plays a somewhat different role than $\mathbf{t}_e$ and $\mathbf{s}_e$, we define $\mathbf{j}_\phi$ by analogy to (2.3)

$$\mathbf{j}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{j}_e \tag{2.6}$$

In the present paper, we shall establish that the various relations given above are not just valid for the abelian case. They also apply in the non-abelian case, under the condition that a formulation is used in which the pseudoparticles do not carry charge or spin (see Section 4). We shall see that in all the cases we consider, such a formulation can indeed be given.

The other important role the K-matrices play will be described in the next section, namely the role as statistics matrix in the sense of the Haldane exclusion statistics of the (quasi) particles. Also, we will explain a notion of "duality" which is important in this context, and rederive some of the relations given above.

3. ABELIAN EXCLUSION STATISTICS

An important consequence of the concept of an "ideal gas of fractional statistics particles" is the notion of 1-particle distribution functions which generalize the familiar Fermi–Dirac and Bose–Einstein distributions. These distributions can be derived from "1-particle grand canonical partition functions." These quantities, which we denote by λ_i, satisfy the following set of equations, which were independently derived by Isakov, Dasnières de Veigy–Ouvry and Wu (IOW)[19]

$$\left(\frac{\lambda_i - 1}{\lambda_i}\right) \prod_j \lambda_j^{\mathbf{K}_{ij}} = z_i \tag{3.1}$$

where $\lambda_i = \lambda_i(z_1,..., z_n)$, with $z_i = e^{\beta(\mu_i - \varepsilon)}$ the generalized fugacity of species i. Note that the energy ε may also include contributions from the coupling of the charge and spin of the quasiparticles to external electric and magnetic fields. Hence the information about charge and spin of the quasiparticles is also encoded in these generalized fugacities. The fugacities of the particles will be important for the distinction between abelian and non-abelian statistics, as we will point out later. The matrix **K** is the so-called "statistics matrix" and describes, at least in the original situation in which Haldane introduced his new notion of statistics, the statistical interaction of particles of different species.

From the solutions λ_i of the IOW-equations (3.1) the one-particle distribution functions $n_i(\varepsilon)$ are obtained as

$$n_i(\varepsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j \lambda_j|_{z_i = e^{\beta(\mu_i - \varepsilon)}} = \sum_j z_j \frac{\partial}{\partial z_j} \log \lambda_i|_{z_i = e^{\beta(\mu_i - \varepsilon)}} \tag{3.2}$$

where we have assumed that the matrix **K** is symmetric.

The relation between, on the one hand, the K-matrix of an abelian quantum Hall fluid and, on the other hand, the exclusion statistics of its charged edge excitations, can be described as follows. The charged edge excitations are described by a specific Conformal Field Theory (CFT), also known as a chiral Luttinger liquid. Following a procedure first proposed in ref. 20, one may associate a notion of fractional exclusion statistics to a set of fundamental excitations in this CFT. Selecting a particular set of negatively charged "electron type" excitations together with a "dual" set of positively charged quasihole excitations, one precisely finds fractional exclusion statistics in the sense of Haldane, with statistics matrix **K** given by

$$\mathbf{K} = \mathbf{K}_e \oplus \mathbf{K}_\phi \tag{3.3}$$

with $\mathbf{K}_e$ and $\mathbf{K}_\phi$ the K-matrices for the abelian quantum Hall state. For the principal Laughlin series at filling fraction $\nu = 1/m$, this result was obtained in ref. 6, in its general form it first appeared in our paper.[14] The relation of the identification (3.3) with character identities involving so called Universal Chiral Partition Functions will be discussed in Section 5.

In ref. 7, a slightly different identification between the K-matrix and a statistics matrix, amounting to $\mathbf{K}=\mathbf{K}_e$, was proposed. The two proposals can be reconciled by realizing that we, in our analysis of edge excitations, restrict ourselves to quanta of positive energy only. From the duality relations that we discuss below, one learns that, in a precise sense, quasihole quanta of positive energy can be traded for holes in a "Fermi sea" of electron-type quanta at negative energy, and in this way one arrives at a complete description in terms of the matrix $\mathbf{K}_e$ alone.

One of the main themes in this paper will be the identification of statistics matrices $\mathbf{K}$ for excitations over non-abelian quantum Hall states. Extending the identification (3.3) to the non-abelian case, we shall propose K-matrices for the non-abelian quantum Hall states. We would like to stress that, although many of the formulas from the well known abelian K-matrix description still hold for the generalized K-matrices we find here, the description for the non-abelian states is on an entirely different footing. The abelian K-matrices were introduced to describe quantum Hall states in the "most general" way, i.e., by trying to implement the hierarchical schemes in a general way. In the non-abelian case, we need the K-matrix structure to keep track of the non-abelian statistics. So although we use a matrix structure, we are not describing a hierarchical situation.

We continue this section with a discussion of the fundamental "particle-hole" duality between the electron and the quasihole sectors of the theory. To show how this duality works, we assume that we have n quasiholes ϕ and n electron-like particles Ψ described by the matrices $\mathbf{K}_\phi$ and $\mathbf{K}_e$, respectively. We assume that (i) $\mathbf{K}_\phi=\mathbf{K}_e^{-1}$, and (ii) there is no mutual exclusion statistics between the two sectors (meaning that the statistics matrix is given by the direct sum (3.3)). These two conditions in fact constitute what we mean by duality in this context. In the context of low-energy effective actions for abelian fqH systems, a similar notion of duality has been considered (see, e.g., ref. 17 and references therein).

With the matrices $\mathbf{K}_\phi$ and $\mathbf{K}_e$, two independent systems of IOW-equations can be written down, and these systems are related by the duality (for clarity, we will denote the single level partition function for the quasiholes and electron-like particles by λ_i and μ_i respectively; the corresponding fugacities will be denoted by x_i and y_i)

$$\lambda_i=\frac{\mu_i}{\mu_i-1}, \qquad x_i=\prod_j y_j^{-(\mathbf{K}_e)_{ij}^{-1}} \tag{3.4}$$

as can be verified easily.

As an illustration of the duality, we calculate the central charge of the conformal field theory that describes the edge excitations. We focus on the abelian case. In the non-abelian case, which we discuss in the next section, there will be a subtraction term due to the presence of pseudoparticles.

In general, for abelian quantum Hall states, the central charge c_{CFT} is given by

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2} \int_0^1 \frac{dz}{z} \log \lambda_{\mathrm{tot}}(z) \tag{3.5}$$

where $\lambda_{\mathrm{tot}}(z)$ denotes the product $\prod_j \lambda_j$ evaluated at $z_j = z$ for all j. It has been shown (see refs. 13, 12 and references therein), that this can be rewritten in the following form

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2} \sum_i L(\xi_i) \tag{3.6}$$

where $L(z)$ is Rogers' dilogarithm

$$L(z) = -\frac{1}{2} \int_0^z dy \left(\frac{\log y}{1-y} + \frac{\log(1-y)}{y} \right) \tag{3.7}$$

The quantities ξ_i which appear in Eq. (3.6) are solutions to the central charge equations

$$\xi_i = \prod_j (1-\xi_j)^{\mathbf{K}_{ij}} \tag{3.8}$$

For the abelian quantum Hall case, we have two matrices $\mathbf{K}_\phi$ and $\mathbf{K}_e$ and we need the solutions ξ_i and η_i of the equations

$$\xi_i = \prod_{j=1}^n (1-\xi_j)^{(\mathbf{K}_\phi)_{ij}}, \qquad \eta_i = \prod_{j=1}^n (1-\eta_j)^{(\mathbf{K}_e)_{ij}} \tag{3.9}$$

By virtue of the duality, these solutions are related by a simple equation: $\eta_i = 1 - \xi_i$. This leads to

$$\sum_i L(\xi_i) + \sum_i L(\eta_i) = \sum_i (L(\xi_i) + L(1-\xi_i)) = nL(1) = n\frac{\pi^2}{6} \tag{3.10}$$

So in the abelian case, we correctly find that the central charge is just given by the number of species in the theory, $c_{\mathrm{CFT}} = n$.

4. NON-ABELIAN EXCLUSION STATISTICS

In this section, we focus on K-matrices and statistics matrices for non-abelian quantum Hall states. We shall first introduce new types of particles, pseudoparticles and composite particles, and explain the role they play in the non-abelian case. We also extend the notion of duality to the non-abelian case. After that we discuss various aspects (filling factors and shift map) of the quantum Hall data **K**, **t**, **s** and **j** in the non-abelian case.

Among the new particles that appear in non-abelian theories are so called "composite" particles in the electron sector. These will show up as particles which have multiple electron charges. We introduce an integer label l_i for an order-l_i composite particle of charge $(\mathbf{t}_e)_i = -l_i$.

In the quasihole sector, we encounter so called pseudoparticles, which do not carry any energy, but rather act as a book-keeping device that keep track of "internal degrees of freedom" of the physical quasiholes. The notion of a "pseudoparticle" can be traced back to so-called string solutions to the Bethe equations for quantum integrable systems in one dimension, such as the Heisenberg XXX chain (see ref. 21, where the contribution to the thermodynamics of the string solutions for the XXX chain is computed). Pseudoparticles were used (and received their name) in the TBA analysis of integrable systems with non-diagonal particle scattering (see, e.g., ref. 22). In the context of exclusion statistics they have been discussed in refs. 7, 11, 12, and 14. We assign the label $l_i = 0$ to all pseudoparticles.

An important observation, first made in ref. 14, is that the duality between the electron and quasihole sectors naturally links the presence of composite particles in one sector to the presence of pseudoparticles in the other. Physically, this is a link between the pairing physics of the non-abelian quantum Hall states and the non-abelian statistics of their fundamental excitations.

4.1. Composites, Pseudoparticles, and Null-Particles

The presence of pseudoparticles and composite particles calls for a slight generalization of the discussion of the previous section. When focusing on the dependence of the λ_i on the energy ε, the natural specialization of the generalized fugacities z_i is given by $z_i = z^{l_i}$, with $z = e^{-\beta\varepsilon}$. In the presence of $l_i \neq 1$, the 1-particle distribution functions take the form [note that a composite particle labeled by ε carries energy $l_i \varepsilon$]

$$n_i(\varepsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j [\lambda_j]^{l_j|_{z_i = e^{\beta(\mu_i - l_i \varepsilon)}}} = \sum_j l_j z_j \frac{\partial}{\partial z_j} \log \lambda_i|_{z_i = e^{\beta(\mu_i - l_i \varepsilon)}} \qquad (4.1)$$

With the following definition of $\lambda_{\text{tot}}(z)$

$$\lambda_{\text{tot}}(z) = \prod_i \left[\lambda_i(z_j = z^{l_j})\right]^{l_i} \tag{4.2}$$

the central charge c_{CFT} is again given by the expression (3.5). We note that in the specialized IOW equations, with $z_i = z^{l_i}$, the right hand side of the equations for pseudoparticles is equal to 1. When focusing on quantum numbers other than energy, such as spin, we will consider slightly more general versions of the quantity λ_{tot}.

In all examples (abelian and non-abelian) that are explicitly discussed in this paper, we assume a choice of particle basis such that $\boldsymbol{l}_e = -\mathbf{t}_e$. For the abelian quantum Hall states we further assume that $(\mathbf{t}_e)_i = -1$ for all i. In the quasihole sector we specify $(\boldsymbol{l}_\phi)_i = (1/q_{\text{qp}})(\mathbf{K}_\phi)_{ij}\,(\boldsymbol{l}_e)_j$, where q_{qp} is the smallest (elementary) charge in the quasihole sector. [This implies that, even in the abelian case, we may treat some of the quasiholes as composites of the most fundamental ones, thereby generalizing the discussion of the previous section.]

Under these assumptions, we find that under duality $\lambda_{\text{tot}}(x)$ and $\mu_{\text{tot}}(y)$ are related in the following way

$$\lambda_{\text{tot}}(x) = x^{\gamma} \mu_{\text{tot}}^{\alpha}(y), \qquad y = x^{-\beta} \tag{4.3}$$

with

$$\alpha = \beta = 1/q_{\text{qp}}, \qquad \gamma = \nu/q_{\text{qp}}^2 \tag{4.4}$$

A clear sign of non-abelian statistics is found in the way the quantity λ_i for physical particles depends on the fugacity z_i. Putting $z_l = 1$ for all pseudoparticles, and focusing on the small z behaviour of λ_i, one finds

$$\lambda_i = 1 + \alpha_i z_i + O(z^2) \tag{4.5}$$

In the abelian case, $\alpha_i = 1$, whereas in the non-abelian case $\alpha_i > 1$. The factors α_i lead to multiplicative factors in the Boltzmann tails of the one-particle distribution functions for physical particles. The quantities α_i are in fact the largest eigenvalues of the fusion matrix,[13] i.e., the quantum dimensions[23] of the conformal field theory associated to the quantum Hall state, and can easily be calculated for the cases we deal with (see Sections 6 and 7.2).

In ref. 14, we presented a generalized K-matrix structure for some recently proposed quantum Hall states. The proposed K-matrices were identified via their role as statistics matrices for the fundamental charged edge excitations. In the quasihole sector, the non-abelian statistics leads to a specific set of pseudoparticles and an associated statistics matrix $\mathbf{K}_\phi$.[11, 12] The matrix $\mathbf{K}_e$, related to $\mathbf{K}_\phi$ by the duality $\mathbf{K}_e = \mathbf{K}_\phi^{-1}$, refers to particles which are identified as composites of the fundamental electron-like excitation. From the point of view of the wave functions for the non-abelian quantum Hall states,[2, 24, 10] the presence of composite excitations is very natural. This is because the non-abelian states show a behaviour which is called *clustering* (of order k, where k is a label of the states[24, 10]). This order-k clustering means that up to k particles can come to the same position, without making the wave function zero, whereas, as soon as $k+1$ particles are located at the same positions, the wave function becomes identically zero. In refs. 25 and 14 it was argued that the wave functions which show pairing (at $k=2$), are related (in the non-magnetic limit, i.e., in the limit of $\nu \to \infty$) to BCS superconductivity.

Composite particles are identified as particles whose generalized fugacities are specific combinations of the generalized fugacities of other particles, i.e., all quantum numbers of composite particles are completely determined in terms of the quantum numbers of their constituents. It has been shown in ref. 12 that particular kinds of composite particles, so-called null-particles, accounting for the null-states in the quasiparticle Fock spaces, are often needed to interpret the system in terms of Haldane's exclusion statistics or, equivalently, to write the partition function in UCPF form (see also Section 5.2).

We now turn to the computation of the central charge c_{CFT} in the non-abelian case. It was shown in ref. 12, that the presence of pseudoparticles leads to a simple correction term that is subtracted from the abelian result $c_{\mathrm{CFT}} = n$. For the pseudoparticles, a system of equations like Eq. (3.9) can be written down

$$\xi'_i = \prod_j{}' \, (1 - \xi'_j)^{\mathbf{K}_{ij}} \tag{4.6}$$

where the prime indicates that the product is restricted to pseudoparticles. The correction term is given by a sum over the dilogarithm of the solutions of (4.6), leading to

$$c_{\mathrm{CFT}} = n - \frac{6}{\pi^2} \sum_i{}' \, L(\xi'_i) \tag{4.7}$$

4.2. On Filling Factors

Up to now, we merely asserted that the statistics matrices $\mathbf{K}$ can also serve as (generalized) K-matrices for non-abelian quantum Hall states. To make this statement more clear, we will now investigate how some of the "K-matrix results" for abelian quantum Hall states generalize to the non-abelian case. In this derivation, we make the assumption that the pseudoparticles do not carry charge or spin. In all cases that are explicitly considered in Sections 6 and 7 this assumption holds in the simplest formulation. If pseudoparticles do carry spin or charge, the formulas we obtain below need to be modified.

Let us start with the filling factor corresponding to state which is described by the IOW-equations, for a statistics matrix $\mathbf{K}_e$, charge vector $\mathbf{t}_e$, and labels $\boldsymbol{l}_e = -\mathbf{t}_e$. We couple the system to an electric field by taking $y_i = y^{-(\mathbf{t}_e)_i}$. [This is when the orientation of the electric field is such that the response is carried by the negatively charged excitations.] The large y (i.e., low temperature) behaviour of the IOW-equations (3.1) is then given by the following set of relations

$$\prod_j \mu_j^{(\mathbf{K}_e)_{ij}} \sim y^{-(\mathbf{t}_e)_i} \tag{4.8}$$

which imply, when $\mathbf{K}$ is symmetric (which is assumed throughout the paper) and invertible

$$\mu_{\text{tot}} = \prod_i \mu_i^{-(\mathbf{t}_e)_i} \sim y^{\mathbf{t}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{t}_e} \tag{4.9}$$

Because the left hand side of Eq. (4.9) in the $T \to 0$ limit determines the filling factor ν through $\mu_{\text{tot}} \sim y^{\nu}$, we find the well-known formula

$$\nu = \mathbf{t}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{t}_e \tag{4.10}$$

For the opposite orientation of the electric field, a similar expression is obtained by starting from the K-matrix for the (positively charged) quasiholes

$$\nu = \mathbf{t}_\phi \cdot \mathbf{K}_\phi^{-1} \cdot \mathbf{t}_\phi \tag{4.11}$$

This result could also have been obtained by using Eq. (4.10) and the transformation properties of $\mathbf{K}_e$ and $\mathbf{t}_e$ under duality. We remark that the above derivations explicitly assume that only the physical particles respond to the electric field, i.e., that all pseudoparticles are neutral.

Let us now turn to the spin Hall conductance, and the corresponding spin filling factor. The derivation of the corresponding spin filling factor

$$\sigma = \mathbf{s}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{s}_e \tag{4.12}$$

goes along the same lines as the derivation of the electron filling factor. As an extra step, one needs to relate the fugacities of the spin up and down particles by $y_\uparrow = 1/y_\downarrow = z$. This results in

$$\prod_i \mu_i^{(\mathbf{s}_e)_i} \sim z^{\mathbf{s}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{s}_e} \tag{4.13}$$

leading to Eq. (4.12). It is important to note that this formula only holds in the cases where the pseudoparticles in the ϕ-sector do not carry spin. As a check on this formula, one would like to have a procedure to obtain the spin filling factor directly from the wave functions, as is possible for the electron filling factor. To do this, one has to count the zeros of the wave function with respect to one reference particle (of a given spin, say, up). The total number of zeros gives the total flux needed on the sphere as a linear function of the total number of electrons N_e. By using the relation between N_e and N_Φ given in (2.4) one obtains the electron filling factor and the shift. To obtain the spin filling factor, one has to keep track of two different types of zeros, namely those with respect to a particle of the same spin, and the ones with respect to particles of the other spin. We will denote the number of these zeros by $N_\Phi^\uparrow$ and $N_\Phi^\downarrow$ respectively. The electron and spin filling factors are obtained from

$$\begin{aligned} N_\Phi &= N_\Phi^\uparrow + N_\Phi^\downarrow = \frac{1}{\nu} N_e - \mathcal{S} \\ N_\Phi^\uparrow - N_\Phi^\downarrow &= \frac{1}{\sigma} N_e - \mathcal{S} \end{aligned} \tag{4.14}$$

We applied this procedure to the non-abelian spin singlet states of ref. 10 (the explicit form of the wave functions will be given elsewhere[26]), and indeed found the same results for the electron and spin filling factor as obtained from the K-matrix formalism, Eq. (7.1). Also the electron filling factor for the Read–Rezayi states is reproduced correctly, see Eq. (6.1). In addition, for both types of states we found that the shift on the sphere is in agreement with (2.5) for $g = 0$.

Summarizing, we have presented evidence that duality relations

$$\mathbf{K}_\phi = \mathbf{K}_e^{-1}, \qquad \mathbf{t}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{t}_e, \qquad \mathbf{s}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{s}_e, \qquad \mathbf{j}_\phi = -\mathbf{K}_e^{-1} \cdot \mathbf{j}_e \tag{4.15}$$

are applicable to both abelian and non-abelian quantum Hall states, and that the expressions (2.2) for the filling factors ν and σ apply to the non-abelian case, in a formulation where pseudoparticles do not carry spin or charge.

4.3. Shift Map

Suppose we have a fractional quantum Hall system described by the data $(\mathbf{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$. We can then construct a family of fractional quantum Hall systems, parametrized by $M \in \mathbb{Z}_+$, by applying the "shift map" $\mathscr{S}_M$ introduced in ref. 27. In the cases we consider, M odd (even) corresponds to a fermionic (bosonic) state respectively. At the level of wave functions $\Psi(z)$, $\mathscr{S}_M$ simply acts as a multiplicative Laughlin factor $\prod_{i<j}(z_i - z_j)^M$. Thus, $\mathscr{S}_M$ increases the number of flux quanta by

$$N_\Phi \mapsto N_\Phi + M(N_e - 1) = \left(\frac{1}{\nu} + M\right) N_e - (\mathscr{S} + M) \tag{4.16}$$

i.e.,

$$\nu^{-1} \mapsto \nu^{-1} + M, \qquad \sigma \mapsto \sigma, \qquad \mathscr{S} \mapsto \mathscr{S} + M \tag{4.17}$$

In fact, $\mathscr{S}_M$ acts on the fqH data $(\mathbf{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$ as

$$\begin{aligned}
\mathscr{S}_M \mathbf{K}_e &= \mathbf{K}_e + M \mathbf{t}_e \mathbf{t}_e \\
\mathscr{S}_M \mathbf{t}_e &= \mathbf{t}_e \\
\mathscr{S}_M \mathbf{s}_e &= \mathbf{s}_e \\
\mathscr{S}_M \mathbf{j}_e &= \mathbf{j}_e + \frac{M}{2} \mathbf{t}_e
\end{aligned} \tag{4.18}$$

One easily checks that (4.18), together with (4.10), leads to the shift in ν^{-1} as given in (4.17). By duality (4.15) one obtains

$$\begin{aligned}
\mathscr{S}_M \mathbf{K}_\phi &= \mathbf{K}_\phi - \frac{M}{1+\nu M} \mathbf{t}_\phi \mathbf{t}_\phi \\
\mathscr{S}_M \mathbf{t}_\phi &= \frac{1}{1+\nu M} \mathbf{t}_\phi \\
\mathscr{S}_M \mathbf{s}_\phi &= \mathbf{s}_\phi \\
\mathscr{S}_M \mathbf{j}_\phi &= \mathbf{j}_\phi - \frac{M}{2} \left(\frac{\nu \mathscr{S} - 1}{1+\nu M}\right) \mathbf{t}_\phi
\end{aligned} \tag{4.19}$$

A few remarks should be made. By using the duality (4.15), one actually finds for the action of the shift map on $\mathbf{s}_\phi$: $\mathcal{S}_M \mathbf{s}_\phi = \mathbf{s}_\phi + (M(\mathbf{t}_\phi \cdot \mathbf{s}_e)/(1+\nu M))\,\mathbf{t}_\phi$. However, the shift map is only supposed to act on the charge component of the particles, thus we would like to demand that $\mathcal{S}_M \mathbf{s}_\phi = \mathbf{s}_\phi$. Therefore, for consistency, we *require*

$$\mathbf{t}_\phi \cdot \mathbf{s}_e = -\mathbf{t}_e \cdot \mathbf{K}_e^{-1} \cdot \mathbf{s}_e = 0 \tag{4.20}$$

leading to (4.19). Of course, relation (4.20) is just the statement that for spin singlet states there should be a $\mathbb{Z}_2$ symmetry $(\mathbf{t}_e, \mathbf{s}_e) \mapsto (\mathbf{t}_e, -\mathbf{s}_e)$. Equation (4.20) is fulfilled for all our examples (if we take $\mathbf{s}_e = 0$ for the spin polarized states). Although, in general, $\mathbf{j}_e$ has to be treated as an independent variable, for the examples discussed in Sections 6 and 7 all formulas are consistent with the relation $\mathbf{j}_e = \mathbf{s}_e + (\mathcal{S}/2(1-g))\,\mathbf{t}_e$.

In this paper we will be mainly concerned with fractional quantum Hall systems corresponding to conformal field theories $\hat{\mathfrak{g}}_{k,M}$ which are deformations of the conformal field theory based on the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level k. The $\hat{\mathfrak{g}}$-symmetry greatly simplifies the determination of the fqH data $(\mathbf{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$ for $\hat{\mathfrak{g}}_k$. The fqH data for $(\hat{\mathfrak{g}})_{k,M}$ are then simply obtained by applying the shift operator $\mathcal{S}_M$ as in (4.18). The action of the shift map can be visualized as follows. Charge is usually identified with a particular direction in the weight lattice of $\mathfrak{g}$. The degrees of freedom associated to this direction can be represented by a chiral boson compactified on a circle of some radius R. The shift map $\mathcal{S}_M$ has the effect of rescaling the radius R while keeping all other directions in the weight diagram fixed.

4.4. Composites

The description of a physical system in terms of a set of n quasiparticles with mutual exclusion statistics given by a matrix $(\mathbf{K}_{ij})_{i \leqslant i,\, j \leqslant n}$ is not unique. In particular one may extend the number of quasiparticles by introducing composites as we will now explain.

Consider the IOW-equations (3.1) with

$$\mathbf{K} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \tag{4.21}$$

If we define the operation $\mathscr{C}_{ij}$, corresponding to adding a composite of the quasiparticles i and j to the system, by

$$\mathscr{C}_{ij}\mathbf{K}=\left(\begin{array}{cccc:c} a_{11} & \cdots & & a_{1n} & a_{1i}+a_{1j} \\ \vdots & & & \vdots & \vdots \\ & & a_{ij}+1 & & \\ & a_{ji}+1 & & & \\ a_{n1} & \cdots & & a_{nn} & a_{ni}+a_{nj} \\ \hdashline a_{i1}+a_{j1} & \cdots & & a_{in}+a_{jn} & a_{ii}+2a_{ij}+a_{jj} \end{array}\right) \tag{4.22}$$

and

$$\mathscr{C}_{ij}\mathbf{z}=(\mathbf{z}_1,..., \mathbf{z}_n; \mathbf{z}_i\mathbf{z}_j) \tag{4.23}$$

such that, in particular,

$$\begin{aligned} \mathscr{C}_{ij}\mathbf{t}&=(\mathbf{t}_1,..., \mathbf{t}_n; \mathbf{t}_i+\mathbf{t}_j) \\ \mathscr{C}_{ij}\mathbf{s}&=(\mathbf{s}_1,..., \mathbf{s}_n; \mathbf{s}_i+\mathbf{s}_j) \end{aligned} \tag{4.24}$$

then the two systems are equivalent, at least at the level of thermodynamics. The solutions $\{\lambda_i\}$ to the IOW-equations defined by $(\mathbf{K}, \mathbf{z})$ and $\{\lambda'_i\}$ defined by $(\mathbf{K}', \mathbf{z}')=(\mathscr{C}_{ij}\mathbf{K}, \mathscr{C}_{ij}\mathbf{z})$ are simply related by

$$\begin{aligned} \lambda'_i&=\frac{\lambda_i+\lambda_j-1}{\lambda_j}, \qquad \lambda'_j=\frac{\lambda_i+\lambda_j-1}{\lambda_i}, \\ \lambda'_{n+1}&=\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j-1}, \qquad \lambda'_k=\lambda_k, \quad (k\neq i, j, n+1) \end{aligned} \tag{4.25}$$

Note that, in particular, it follows $\lambda_i=\lambda'_i\lambda'_{n+1}$ and $\lambda_j=\lambda'_j\lambda'_{n+1}$ such that $\lambda_{\text{tot}}=\lambda'_{\text{tot}}$. Also, from $\lambda_i=\lambda'_i\lambda'_{n+1}$ and $\lambda_j=\lambda'_j\lambda'_{n+1}$ one sees that the original one-particle partition functions for i and j, receive contributions from the new particles i and j, respectively, as well as from the composite particle $n+1$. The operation $\mathscr{C}_{ij}$ has the effect that states in the spectrum containing both particles i and j get less dense (their mutual exclusion statistics is bumped up by 1), while the resulting "gaps" are now filled by the new composite particle.

A consistency check on the equivalence of the systems described by $(\mathbf{K}, \mathbf{z})$ and $(\mathbf{K}', \mathbf{z}')$ is the fact that both lead to the same central charge as a consequence of the five-term identity for Rogers' dilogarithm (see ref. 12).

Finally, note that the shift map $\mathscr{S}_M$ of Eq. (4.18) and composite operation $\mathscr{C}_{ij}$ of Eqs. (4.22) and (4.24) commute, i.e.,

$$\mathscr{S}_M \mathscr{C}_{ij} = \mathscr{C}_{ij} \mathscr{S}_M \tag{4.26}$$

as one would expect.

5. THE UCPF AND EXCLUSIONS STATISTICS

5.1. Quasiparticle Basis and Truncated Partition Function

Quasiparticles in two dimensional conformal field theories are represented by so-called chiral vertex operators $\phi^{(i)}(z)$ that intertwine between the irreducible representations of the chiral algebra. Given a set of quasiparticles $\phi^{(i)}(z)$, $i = 1,..., n$, one has to determine a basis for the Fock space created by the modes $\phi^{(i)}_{-s}$, i.e., a maximal, linearly independent set of vectors

$$\phi^{(i_N)}_{-s_N} \cdots \phi^{(i_2)}_{-s_2} \phi^{(i_1)}_{-s_1} |\omega\rangle \tag{5.1}$$

with suitable restrictions on the mode sequences $(s_1,..., s_N)$ (which may depend on the "fusion paths" $(i_1,..., i_N)$), as well as a set of vacua $|\omega\rangle$ (see refs. 13 and 12 for more details). The partition function $Z(\mathbf{z}; q)$ is then defined by

$$Z(\mathbf{z}; q) = \mathbf{Tr}\left(\left(\prod_i z_i^{N_i}\right) q^{L_0}\right) \tag{5.2}$$

where the trace is taken over the basis (5.1) and N_i denotes the number operator for quasiparticles of type i while $L_0 = \sum_i s_i$ for a state of type (5.1). During this discussion on the UCPF, we use the following, in the literature standard notation $q = e^{-\beta\varepsilon_0}$, where ε_0 is some fixed energy scale, and $z_i = e^{\beta\mu_i}$.

Exclusion statistics in conformal field theory can be studied by means of recursion relations for truncated partition functions.[20] Truncated partition functions $P_{\mathbf{L}}(\mathbf{z}; q)$, for $\mathbf{L} = (L_1,..., L_n)$, are defined by taking the partition function of those states (5.1) where all the modes s for quasiparticles of species i satisfy $s \leqslant L_i$. By definition, for large $\mathbf{L}$, we will have (see refs. 13 and 12 for more details)

$$P_{\mathbf{L}+\mathbf{e}_i}(z; q)/P_{\mathbf{L}}(z; q) \sim \lambda_i(z_i q^{L_i}) \tag{5.3}$$

where $\mathbf{e}_i$ denotes the unit vector in the i-direction. In particular, if the generalized fugacities z_i are given by $z_i = z^{l_i}$, for some fixed z, and the quasiparticle modes are truncated by $L_i = l_i L$, then we find, using (4.2)

$$P_{L+1}(z; q)/P_L(z; q) \sim \lambda_{\text{tot}}(zq^L) \tag{5.4}$$

where $P_L(z; q) = P_{l_1 L, l_2 L, \ldots, l_n L}(z_i = z^{l_i}; q)$. Thus, given a set of recursion relations for the truncated partition functions $P_{\mathbf{L}}(z; q)$, one derives algebraic equations for the one-particle partition functions $\lambda_i(z)$ by taking the large $\mathbf{L}$ limit. In particular one can find an equation for $\lambda_{\text{tot}}(z)$ from $P_L(z; q)$ by using (5.4). For all conformal field theories that have been studied this way it turns out that one finds agreement between these λ-equations and the IOW-equations (3.1) corresponding to a specific statistics matrix $\mathbf{K}$ (see, in particular, ref. 13).

5.2. The Universal Chiral Partition Function

Based on many examples, it has become clear that the characters of the representations of all conformal field theories can be written in the form of, what is now known as, a universal chiral partition function (UCPF) (see in particular, ref. 28 and references therein)

$$Z(\mathbf{K}; \mathbf{Q}, \mathbf{u} \mid \mathbf{z}; q) = \sum_{\mathbf{m}}{}' \left(\prod_i z_i^{m_i} \right) q^{\frac{1}{2}\mathbf{m} \cdot \mathbf{K} \cdot \mathbf{m} + \mathbf{Q} \cdot \mathbf{m}} \prod_i \begin{bmatrix} ((\mathbb{1} - \mathbf{K}) \cdot \mathbf{m} + \mathbf{u})_i \\ m_i \end{bmatrix} \tag{5.5}$$

where $\mathbf{K}$ is a (rational) $n \times n$ matrix, $\mathbf{Q}$ and $\mathbf{u}$ are certain n-vectors and the sum over $m_1, \ldots, m_n$, is over the nonnegative integers subject to some restrictions (which, throughout this paper, are taken to be such that the coefficients in the q-binomials are integer). The q-binomial (Gaussian polynomial) is defined by

$$\begin{bmatrix} M \\ m \end{bmatrix} = \frac{(q)_M}{(q)_m (q)_{M-m}}, \qquad (q)_m = \prod_{k=1}^{m} (1 - q^k) \tag{5.6}$$

The vectors $\mathbf{Q}$ and $\mathbf{u}$ as well as the restrictions on the summation variables, will in general depend on the particular representation of the conformal field theory, while $\mathbf{K}$ is independent of the representation. To write the conformal characters in the form (5.5) may require introducing null-quasiparticles which account for null-states in the quasiparticle Fock space.[12] The null-quasiparticles are certain composites, hence their fugacities z_i in (5.5) are specific combinations of the fugacities of their constituents.

It has been conjectured that the UCPF (5.5) is precisely the partition function (5.2) of a set of quasiparticles with exclusion statistics given by the same matrix $\mathbf{K}$, where $u_i = \infty$ corresponds to a physical quasiparticle and $u_i < \infty$ to a pseudoparticle.[11, 12] This conjecture has been verified in numerous examples (see refs. 11 and 12 for references). A convincing piece of evidence in support of this conjecture is the fact that the asymptotics of the character (5.5) (in the thermodynamic limit $q \to 1^-$) is given by exactly the same formula as the one for the IOW-equations[12] (see also refs. 29 and 30 for $z_i = 1$). In the next section we establish the correspondence in a more direct way.

For future convenience let us introduce the limiting form of the UCPF (5.5) when all $u_i \to \infty$, i.e., the case that all quasiparticles are physical and the exclusion statistics is abelian

$$Z_\infty(\mathbf{K}; \mathbf{Q}) = \sum_{\mathbf{m}}{}' \left(\prod_i z_i^{m_i} \right) \frac{q^{\frac{1}{2}\mathbf{m} \cdot \mathbf{k} \cdot \mathbf{m} + \mathbf{Q} \cdot \mathbf{m}}}{\prod_i (q)_{m_i}} \tag{5.7}$$

Note that the limiting UCPFs (5.7) are not all independent, but satisfy (see ref. 31)

$$Z_\infty(\mathbf{K}; \mathbf{Q}) = Z_\infty(\mathbf{K}; \mathbf{Q} + \mathbf{e}_i) + z_i q^{\frac{1}{2}\mathbf{K}_{ii} + \mathbf{Q}_i} Z_\infty(\mathbf{K}; \mathbf{Q} + \mathbf{K} \cdot \mathbf{e}_i) \tag{5.8}$$

as a consequence of

$$\frac{1}{(q)_m} = \frac{q^m}{(q)_m} + \frac{1}{(q)_{m-1}} \tag{5.9}$$

5.3. Relation to Exclusion Statistics

The relation between the UCPF and exclusion statistics can be made more explicit as follows. Suppose the truncated partition functions $P_{\mathbf{L}}(\mathbf{z}; q)$ are given by “finitized UCPFs” of the form

$$P_{\mathbf{L}}(\mathbf{z}; q) = \sum_{\mathbf{m}}{}' \left(\prod_i z_i^{m_i} \right) q^{\frac{1}{2}\mathbf{m} \cdot \mathbf{K} \cdot \mathbf{m} + \mathbf{Q} \cdot \mathbf{m}} \prod_i \begin{bmatrix} (\mathbf{L} + (\mathbb{1} - \mathbf{K}) \cdot \mathbf{m} + \mathbf{u})_i \\ m_i \end{bmatrix} \tag{5.10}$$

for some vectors $(\mathbf{Q}, \mathbf{u})$. Of course, the number of parameters in this expression is overdetermined. Usually we think of $\mathbf{u}$ as being fixed while the meaning of the parameters $\mathbf{L}$ are determined by the cut-off scale. We can of course absorb the $\mathbf{u}$ by shifts in $\mathbf{L}$ (in fact, in practice we often make shifts in the definition of $\mathbf{L}$ to simplify the recursion relations). We also remark that we have introduced finitization parameters L_i also for the pseudoparticles in (5.10) to facilitate deriving recursion relations. In making the identification with the truncated partition functions these parameters are kept at a fixed (usually "small" or even zero) value.

Using

$$\begin{bmatrix} M \\ m \end{bmatrix} = \begin{bmatrix} M-1 \\ m \end{bmatrix} + q^{M-m} \begin{bmatrix} M-1 \\ m-1 \end{bmatrix} \tag{5.11}$$

we find that $P_{\mathbf{L}}(\mathbf{z}; q)$ satisfies the system of recursion relations

$$P_{\mathbf{L}}(\mathbf{z}; q) = P_{\mathbf{L}-\mathbf{e}_i}(\mathbf{z}; q) + z_i q^{-\frac{1}{2}\mathbf{K}_{ii} + \mathbf{Q}_i + \mathbf{u}_i + \mathbf{L}_i} P_{\mathbf{L}-\mathbf{K}\cdot\mathbf{e}_i}(\mathbf{z}; q) \tag{5.12}$$

Upon dividing by $P_{\mathbf{L}}(\mathbf{z}; q)$, setting $q=1$, taking the large $\mathbf{L}$ limit, and using (5.3), we obtain

$$1 = \lambda_i^{-1} + z_i \prod_j \lambda_j^{-\mathbf{K}_{ji}} \tag{5.13}$$

which are equivalent to the IOW-equations (3.1) with statistics matrix $\mathbf{K}$.

Moreover, for any polynomial $P_{\mathbf{L}}(\mathbf{z}; q)$ satisfying the recursion relation (5.12), the polynomial

$$Q_{\mathbf{L}}(\mathbf{z}; q) = \left(\prod_i z_i^{-L_i} \right) q^{1/2 \mathbf{L}\cdot\mathbf{K}\cdot\mathbf{L} + (\mathbf{Q}+\mathbf{u})\cdot\mathbf{L}} P_{\mathbf{K}\cdot\mathbf{L}}(\mathbf{z}; q^{-1}) \tag{5.14}$$

satisfies the recursion relations (5.12) with dual data $(\mathbf{K}'; \mathbf{Q}', \mathbf{u}', \mathbf{z}')$, given by (cf. (3.4))

$$\mathbf{K}' = \mathbf{K}^{-1}, \qquad \mathbf{Q}' + \mathbf{u}' = \mathbf{K}^{-1} \cdot (\mathbf{Q} + \mathbf{u}), \qquad z_i' = \prod_j z_j^{-\mathbf{K}_{ij}^{-1}} \tag{5.15}$$

Thus, under the assumption that the set of finitized UCPFs (5.10), for fixed $\mathbf{Q}+\mathbf{u}$, form a complete set of solutions to (5.12), the dual polynomial $Q_{\mathbf{L}}(\mathbf{z}', q)$ of (5.14) can again be written as a (finite) linear sum of finitized UCPFs with dual data (5.15). Moreover, by taking the large $\mathbf{L}$ limit of (5.14), using Eqs. (5.3) and (5.13), one recovers the duality relations (3.4) and (4.3).

The above calculation shows that, for quasiparticles whose truncated partition function is given by an expression of the form (5.10), the thermodynamics of these quasiparticles is described by Haldane's exclusion statistics with statistics matrix $\mathbf{K}$. Even though many truncated characters are indeed of the form (5.10) (we will encounter various examples in the remainder of this paper) this is not the general situation. However, in examples it turns out that for all recursion relations for truncated characters there is an associated recursion relation, leading to the *same* λ-equation, which does admit a solution of the form (5.10). The true solution to this recursion relation will in general differ from (5.10) by terms of order q^L. In a sense we can talk about the *universality class* of recursion relations as those recursion relations that give rise to the same λ-equations and hence the same exclusion statistics.

5.4. Composites, Revisited

In Section 4.4 we have seen, at the level of thermodynamics (i.e., the IOW-equations), how to introduce composite particles into the system in such a way that the resulting system is equivalent to the original system. Due to the intimate relation of exclusion statistics with the UCPF, explained in Section 5.3, one would expect that a similar construction is possible at the level of the UCPF. Indeed, upon substituting the following polynomial q-identity (see Appendix A for a proof)

$$\begin{bmatrix} M_1 \\ m_1 \end{bmatrix} \begin{bmatrix} M_2 \\ m_2 \end{bmatrix} = \sum_{m \geqslant 0} q^{(m_1 - m)(m_2 - m)} \begin{bmatrix} M_1 - m_2 \\ m_1 - m \end{bmatrix} \begin{bmatrix} M_2 - m_1 \\ m_2 - m \end{bmatrix} \times \begin{bmatrix} M_1 + M_2 - (m_1 + m_2) + m \\ m \end{bmatrix} \tag{5.16}$$

into the UCPF (5.10) at the (i, j)th entry, and subsequently shifting the summation variables $m_i \mapsto m_i + m$, $m_j \mapsto m_j + m$, yields an equivalent UCPF, based on $n+1$ quasiparticles with data $(\mathscr{C}_{ij}\mathbf{K}; \mathscr{C}_{ij}\mathbf{Q}, \mathscr{C}_{ij}\mathbf{u})$ and $\mathscr{C}_{ij}\mathbf{z}$, where

$$\begin{aligned} \mathscr{C}_{ij}\mathbf{Q} &= (\mathbf{Q}_1, \ldots, \mathbf{Q}_n; \mathbf{Q}_i + \mathbf{Q}_j) \\ \mathscr{C}_{ij}\mathbf{u} &= (\mathbf{u}_1, \ldots, \mathbf{u}_n; \mathbf{u}_i + \mathbf{u}_j) \end{aligned} \tag{5.17}$$

while $\mathscr{C}_{ij}\mathbf{K}$ and $\mathscr{C}_{ij}\mathbf{z}$ are defined in Eqs. (4.22) and (4.23), respectively. Various limiting forms of (5.16), relevant to introducing a composite of two physical particles or one physical particle and one pseudoparticle, are given in Appendix A as well.

6. $\mathfrak{sl}_2$: *K*-MATRICES FOR NON-ABELIAN SPIN POLARIZED STATES

In this section we discuss a family of non-abelian spin polarized fractional quantum Hall systems with underlying conformal field theory $(\widehat{\mathfrak{sl}_2})_{k,M}$ and filling factor

$$\nu_{k,M} = \frac{k}{kM+2} \tag{6.1}$$

For $k=2$ these systems, the so-called q-Pfaffians (where now $q=1/\nu=M+1$), were introduced in ref. 2 while the generalizations to $k>2$ were introduced in ref. 24. The system contains a single quasihole ϕ, with charge $1/(kM+2)$ and an electron operator Ψ with charge -1. At the $(\widehat{\mathfrak{sl}_2})_k$-point (i.e., $M=0$) the quasihole operator ϕ has $\mathfrak{sl}_2$-weight $\alpha/2$, where α is the (positive) root of $\mathfrak{sl}_2$ and corresponds to one component of the chiral vertex operator transforming in the spin-1/2 representation ("spinon," see refs. 32–35), while the electron operator has weight $-\alpha$ and corresponds to the current $J_{-\alpha}$. For general M the charge lattice has to be stretched.

The fqH data $(\mathbf{K}_e, \mathbf{t}_e)$ and their duals $(\mathbf{K}_\phi, \mathbf{t}_\phi)$ for $k=1$ (corresponding to the abelian spin polarized Laughlin states with $\nu=1/(M+2)$[36]) were discussed in ref. 6 and for $k=2$ (the q-Pfaffian) in ref. 14. Here we discuss the generalization (see also ref. 11) to arbitrary k, corresponding to the Read–Rezayi states.[24]

As indicated before, we analyze the conformal field theory $(\widehat{\mathfrak{sl}_2})_{k,M}$ by first analyzing the affine Lie algebra point $M=0$ and subsequently applying the shift map to obtain the result for general M.

The exclusion statistics and UCPF for the doublet of spinon operators in $(\widehat{\mathfrak{sl}_2})_k$ were studied in refs. 35, 37, 11, and 12. It turns out that in this case we need $k-1$ additional charge- and spin neutral pseudoparticles. Omitting the negative isospin spinon, we find (see, in particular, refs. 11 and 12)

$$\mathbf{K}_\phi = \left(\begin{array}{cccccc:c} 1 & -\frac{1}{2} & & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & & & -\frac{1}{2} & 1 & & -\frac{1}{2} \\ \hdashline & & & & -\frac{1}{2} & & \frac{1}{2} \end{array}\right), \qquad \mathbf{t}_\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{6.2}$$

leading, with (4.11), to a filling factor of $\nu = k/2$ in accordance with (6.1).

The data for arbitrary M now follow by applying the shift map $\mathcal{S}_M$ of (4.19), i.e.,

$$\mathbf{K}_\phi^M = \mathcal{S}_M \mathbf{K}_\phi = \left(\begin{array}{ccc} \frac{1}{2}\mathbf{A}_{k-1} & \vdots & \\ & \vdots & \\ & \vdots & -\frac{1}{2} \\ \cdots\cdots & \cdots & \cdots\cdots \\ -\frac{1}{2} & \vdots & \frac{(k-1)M+2}{2(kM+2)} \end{array} \right)$$

$$\mathbf{t}_\phi^M = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{kM+2} \end{pmatrix} \tag{6.3}$$

where, in order to simplify the notation, we have introduced the Cartan matrix $\mathbf{A}_{k-1}$ of $\mathfrak{sl}_k$ (cf. (B.3)). One verifies that (4.11) is satisfied. The IOW-equations, determining the exclusion statistics of the quasiholes, can now be explicitly written down. E.g., for the q-Pfaffian ($k=2$) the following equation for λ_{tot} easily follows from (3.1), in agreement with ref. 9

$$(\lambda_{\text{tot}} - 1)(\lambda_{\text{tot}}^{1/2} - 1) = x^2 \lambda_{\text{tot}}^{(3M+2)/(2(M+1))} \tag{6.4}$$

The small x behaviour of λ_{tot} for general k was obtained from the IOW-equations in ref. 13, with the result

$$\lambda_{\text{tot}}(x) = 1 + \alpha_k x + O(x^2), \qquad \alpha_k = 2\cos\left(\frac{\pi}{k+2}\right) \tag{6.5}$$

It was argued that the factors α can also be obtained as quantum dimension of the appropriate CFT. It is easily checked that the small x behaviour of λ_{tot} in (6.4) indeed satisfies (6.5) for $k=2$. Similar equations for λ_{tot} with $k=3, 4$ were given in ref. 13.

To determine the fqH data $(\mathbf{K}_e, \mathbf{t}_e)$ in the electron sector we observe that the electron operator $\Psi(z)$ is identified with $J_{-\alpha}(z)$. By acting with the negative modes of $J_{-\alpha}(z)$ on the lowest weight vector in the lowest energy sector of some integrable highest weight module $L(\Lambda)$ at level k, one obtains what is known as the principal subspace $W(\Lambda)$ of $L(\Lambda)$ (or, rather, the reflected principal subspace). It is known that the character of the principal subspace can be written in the UCPF form[38, 39] (see Appendix B for a brief summary of the results for $(\widehat{\mathfrak{sl}_n})_k$). For $(\widehat{\mathfrak{sl}_2})_k$ this requires, besides the electron operator Ψ itself, clusters of up to k electron operators. The corresponding K-matrix is given by the $k \times k$ matrix $\mathbf{K}_e = 2\mathbf{B}_k$ where $(\mathbf{B}_k)_{ij} = \min(i, j)$ (see (B.4)), while $\mathbf{t}_e = -(1, 2, ..., k)$. Applying the shift map (4.18) thus gives

$$\mathbf{K}_e^M = \begin{pmatrix} M+2 & 2M+2 & \cdots & kM+2 \\ 2M+2 & 2(2M+2) & \cdots & 2(kM+2) \\ \vdots & \vdots & \ddots & \vdots \\ kM+2 & 2(kM+2) & \cdots & k(kM+2) \end{pmatrix}, \qquad \mathbf{t}_e^M = -\begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} \tag{6.6}$$

One easily verifies that the data $(\mathbf{K}_\phi, \mathbf{t}_\phi)$ and $(\mathbf{K}_e, \mathbf{t}_e)$ are indeed related by the duality relations (4.15), and that Eqs. (4.10) and (4.11) are satisfied.

Moreover, the resulting IOW-equations for $\mu_{\text{tot}} = \mu_1 \mu_2^2$ in case of the q-Pfaffian are given by

$$(\mu_{\text{tot}}^{2(M+1)} - y^2)(\mu_{\text{tot}}^{M+1} - y) = \mu_{\text{tot}}^{3M+2} \tag{6.7}$$

which are indeed related to (6.4) by the duality relations (4.3). Explicitly,

$$\lambda_{\text{tot}}(x) = y^{-2} \mu_{\text{tot}}^{2(M+1)}(y), \qquad y = x^{-2(M+1)} \tag{6.8}$$

Finally, in order to show that the quasihole-electron system based on $\mathbf{K} = \mathbf{K}_\phi^M \oplus \mathbf{K}_e^M$, gives a complete description of the $(\widehat{\mathfrak{sl}_2})_{k,M}$ conformal field theory, we have to show that the chiral character of the latter can be written in terms of a (finite) combination of UCPF characters based on $\mathbf{K}_\phi^M \oplus \mathbf{K}_e^M$. This is indeed possible and discussed in Appendix C. Here we suffice to remark that the central charge, related to the asymptotic behaviour of the characters, works out correctly. Indeed, using standard dilogarithm identities one finds with (4.7)

$$c_\phi + c_e = \frac{3k}{k+2} \tag{6.9}$$

which equals the central charge of $(\widehat{\mathfrak{sl}_2})_{k,M}$.

The above description of the Read–Rezayi states has an interesting application, namely the identification of a particle which acts as a supercurrent in the non-magnetic limit. This identification was made in ref. 14, to which we refer for a more detailed discussion. We use the variable $q = 1/\nu = M + k/2$, in terms of which the non-magnetic limit corresponds to

$q \to 0$. In this limit, all the statistics parameters of the largest composite (with charge $-k$), go to zero, while the statistics parameters of the quasihole diverge. This is easily seen when one writes the statistic matrices (6.6) and (6.3) in terms of q. For these quantum Hall states the fundamental flux quantum is h/ke, because of the order-k clustering. Upon piercing a quantum Hall state with this amount of flux, a quasihole with charge e/kq is formed. This follows from the fact that the filling factor is e^2/qh in physical units. For $q \geqslant 1/k$ this is the lowest charge possible and the electron like excitations correspond to multiple insertions of the flux quantum. This situation changes when we take the limit $q \to 0$. Following ref. 14, we take $q = 1/N$, with N a large integer. The largest composite is formed by inserting an amount of flux $-qkh/e = -kh/Ne$, thus a fraction of the flux quantum. The maximal occupation with this particle (in absence of other particles) is $n_{\max} = 1/k^2 q = N/k^2$. Thus the maximal amount of flux that can be screened by this type of composites is $(-kh/Ne)(N/k^2) = -h/ke$, which is precisely the flux quantum. In conclusion we find that in the non-magnetic limit, the largest composite has bosonic statistics, and can screen an amount of flux up to the flux quantum. This clearly resembles the behaviour of the supercurrent in BCS superconductors.

7. $\mathfrak{sl}_3$: *K*-MATRICES FOR NON-ABELIAN SPIN SINGLET STATES

In ref. 10 a family of non-abelian spin singlet (NASS) states $\Psi_{k,M}$ wave functions with filling factors

$$\nu_{k,M} = \frac{2k}{2kM+3}, \qquad \sigma_{k,M} = 2k \tag{7.1}$$

was constructed. The system has two quasihole excitations $\{\phi_\uparrow, \phi_\downarrow\}$ with one unit of up/down spin and charge $1/(2kM+3)$, while the electron operators $\{\Psi_\uparrow, \Psi_\downarrow\}$ have charge -1. The underlying conformal field theory is $(\widehat{\mathfrak{sl}_3})_{k,M}$. In terms of $\mathfrak{sl}_3$-weights the spin and charge assignment in the $M=0$ case is as follows. Denote the positive simple roots of $\mathfrak{sl}_3$ by α_i, $i = 1, 2$ and the remaining positive non-simple root by $\alpha_3 = \alpha_1 + \alpha_2$. Let ε_i, $i = 1, 2, 3$, denote the weights of the fundamental three dimensional irreducible representation **3** of $\mathfrak{sl}_3$ such that $\varepsilon_i \cdot \varepsilon_j = \delta_{ij} - 1/3$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2$, then $\{\phi_\uparrow, \phi_\downarrow\} = \{\phi^{\varepsilon_1}, \phi^{\varepsilon_2}\}$ while $\{\Psi_\uparrow, \Psi_\downarrow\} = \{J_{-\alpha_2}, J_{-\alpha_3}\}$ (See Fig. 1). The charge and spin direction are identified in the $\mathfrak{sl}_3$ weight diagram as indicated in the figure. For other M the analogous picture is obtained by "stretching" the charge axis.

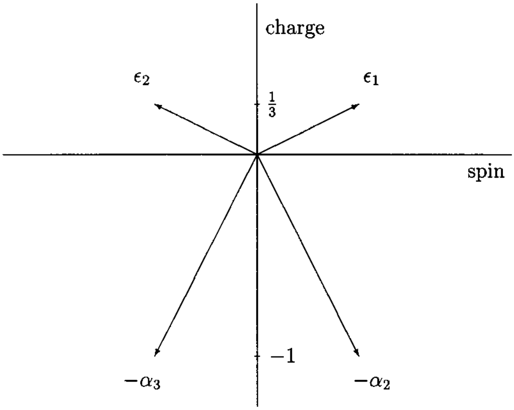

Fig. 1. $\mathfrak{sl}_3$ weight diagram.

In the following sections we analyze the fqH data for the conformal field theory $(\widehat{\mathfrak{sl}_3})_{k,M}$. We first discuss the case $k=1$ (which corresponds to the abelian spin singlet Halperin state with parameters $(M+2, M+2, M+1)^{(40)}$) in some detail and then generalize to the non-abelian case $k>1$.

7.1. $(\widehat{\mathfrak{sl}_3})_{k=1,M}$

The exclusion statistics and UCPF character for the $(\widehat{\mathfrak{sl}_3})_{k=1,M=0}$ conformal field theory, in terms of the quasiparticles $\{\phi^{\varepsilon_1}, \phi^{\varepsilon_2}, \phi^{\varepsilon_3}\}$, were worked out in refs. 41, 20, 13, and 12. Specializing to the subset $\{\phi_\uparrow, \phi_\downarrow\} = \{\phi^{\varepsilon_1}, \phi^{\varepsilon_2}\}$ we have

$$\mathbf{K}_\phi = \tfrac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{t}_\phi = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \qquad \mathbf{s}_\phi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7.2}$$

With (4.11) this leads to $\nu = 2/3$ in agreement with (7.1). Applying the shift map (4.19), the fqH data for $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ are thus given by

$$\mathbf{K}_\phi^M = \mathscr{S}_M \mathbf{K}_\phi = \frac{1}{2M+3}\begin{pmatrix} M+2 & -(M+1) \\ -(M+1) & M+2 \end{pmatrix} \tag{7.3}$$

while

$$\mathbf{t}_\phi^M = \begin{pmatrix} \dfrac{1}{2M+3} \\ \dfrac{1}{2M+3} \end{pmatrix}, \qquad \mathbf{s}_\phi^M = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7.4}$$

The IOW-equation for the total one-particle partition function $\lambda_{\text{tot}} = \lambda_\uparrow \lambda_\downarrow$, resulting from (7.3), is given by

$$\lambda_{\text{tot}} - x_\uparrow x_\downarrow \lambda_{\text{tot}}^{(2M+2)/(2M+3)} - (x_\uparrow + x_\downarrow)\, \lambda_{\text{tot}}^{(M+1)/(2M+3)} - 1 = 0 \tag{7.5}$$

The K-matrix in the electron sector is determined as follows. First of all, the principal subspace of the $(\widehat{\mathfrak{sl}_3})_{k=1,\, M=0}$ integrable highest weight modules is generated by $\{J_{-\alpha_1}, J_{-\alpha_2}\}$ and has a K-matrix given by (see Appendix B)

$$\mathbf{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{7.6}$$

The electron operators $\{\Psi_\uparrow, \Psi_\downarrow\}$, however, are identified with $\{J_{-\alpha_2}, J_{-\alpha_3}\}$. Interpreting $J_{-\alpha_3}$ as the composite $(J_{-\alpha_1} J_{-\alpha_2})$, we can apply the construction of Section 4.4 and find an equivalent K-matrix for the combined $\{J_{-\alpha_1}, J_{-\alpha_2}, J_{-\alpha_3}\}$ system

$$\mathbf{K}' = \mathscr{C}_{12}\mathbf{K} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{7.7}$$

Thus, we conclude that the electron fqH data are given by

$$\mathbf{K}_e = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{t}_e = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{s}_e = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{7.8}$$

And thus, by applying the shift map

$$\mathbf{K}_e^M = \mathscr{S}_M \mathbf{K}_e = \begin{pmatrix} M+2 & M+1 \\ M+1 & M+2 \end{pmatrix}, \qquad \mathbf{t}_e^M = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7.9}$$

Note again that the fqH data in the electron and quasihole sectors, given in Eqs. (7.3), (7.4) and (7.9), are related by the duality (4.15).

The IOW-equation for $\mu_{\text{tot}} = \mu_\uparrow \mu_\downarrow$, resulting from (7.9), is given by

$$\mu_{\text{tot}}^{2M+3} - \mu_{\text{tot}}^{2M+2} - (y_\uparrow + y_\downarrow)\, \mu_{\text{tot}}^{M+1} - y_\uparrow y_\downarrow = 0 \tag{7.10}$$

and is dual to (7.5) in the sense of (4.3). Explicitly,

$$\lambda_{\text{tot}}(x_\uparrow, x_\downarrow) = (y_\uparrow y_\downarrow)^{-1}\, \mu_{\text{tot}}(y_\uparrow, y_\downarrow)^{2M+3} \tag{7.11}$$

where

$$y_\uparrow = x_\uparrow^{-(M+2)} x_\downarrow^{-(M+1)}, \qquad y_\downarrow = x_\uparrow^{-(M+1)} x_\downarrow^{-(M+2)} \tag{7.12}$$

It remains to show that the fqH data $(\mathbf{K}_\phi, \mathbf{t}_\phi, \mathbf{s}_\phi)$ and their duals $(\mathbf{K}_e, \mathbf{t}_e, \mathbf{s}_e)$ give a complete description of the chiral spectrum of the $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ conformal field theory by constructing the $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ characters in terms of (finite) linear combinations of UCPFs based on $\mathbf{K}_e \oplus \mathbf{K}_\phi$. This is delegated to Appendix D. Here we only observe that, since there are no pseudoparticles, Eq. (3.10) immediately gives $c_e + c_\phi = 2$ which is the correct value of the central charge for $(\widehat{\mathfrak{sl}_3})_{k=1,M}$. Note also that c_ϕ and c_e separately depend on M and are, in general, not simple rational numbers, e.g., for $M=0$ we have numerically $c_e = 0.6887$ and $c_\phi = 1.3113$ while for $M \to \infty$ all the central charge is concentrated in the ϕ sector.

Upon generalizing to higher levels $k>1$, it turns out we need an equivalent description of the system described above in terms of three quasihole operators, namely by adding a quasihole operator $\phi^{-\varepsilon_3}$ of $\mathfrak{sl}_3$ weight $-\varepsilon_3$, i.e., of charge 2/3 (for $M=0$) and spinless. The K-matrix for this system can be obtained as a submatrix of the K-matrix describing quasiparticles in the $\mathbf{3} \oplus \mathbf{3}^*$ of $\mathfrak{sl}_3{}^{(12)}$ or, equivalently, by using that $\phi^{-\varepsilon_3}$ is the composite $(\phi^{-\varepsilon_1}\phi^{-\varepsilon_2})^{(41)}$ and using (4.22). We find

$$\mathbf{K}'^{M}_\phi = \mathscr{C}_{12}\mathbf{K}^M_\phi = \frac{1}{2M+3}\begin{pmatrix} M+2 & M+2 & 1 \\ M+2 & M+2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad \mathbf{t}'^{M}_\phi = \begin{pmatrix} \frac{1}{2M+3} \\ \frac{1}{2M+3} \\ \frac{2}{2M+3} \end{pmatrix} \tag{7.13}$$

In the electron sector we can similarly introduce the composite $(J_{-\alpha_2} J_{-\alpha_3})$ and obtain

$$\mathbf{K}'^{M}_e = \mathscr{C}_{12}\mathbf{K}^M_e = \begin{pmatrix} M+2 & M+2 & 2M+3 \\ M+2 & M+2 & 2M+3 \\ 2M+3 & 2M+3 & 4M+6 \end{pmatrix}, \qquad \mathbf{t}'_e = -\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{7.14}$$

Now we observe a curiosity; while obviously the fqH data (7.13) and (7.14) are dual, since they are equivalent to the dual systems given in (7.3) and (7.9), they are not related by the duality transformation (4.15) because both $\mathbf{K}_\phi$ and $\mathbf{K}_e$ are not invertible. The equivalence can also be observed at the level of the resulting IOW-equations which are now given by

$$\begin{aligned} &(\lambda_{\text{tot}}^{1/(2M+3)} - x_\uparrow x_\downarrow)(\lambda_{\text{tot}} - x_\uparrow x_\downarrow \lambda_{\text{tot}}^{(2M+2)/(2M+3)} \\ &\qquad - (x_\uparrow + x_\downarrow)\,\lambda_{\text{tot}}^{(M+1)/(2M+3)} - 1) = 0 \\ &(\mu_{\text{tot}}^{2M+3} - y_\uparrow y_\downarrow)(\mu_{\text{tot}}^{2M+3} - \mu_{\text{tot}}^{2M+2} - (y_\uparrow + y_\downarrow)\,\mu_{\text{tot}}^{M+1} - y_\uparrow y_\downarrow) = 0 \end{aligned} \tag{7.15}$$

Because of the first factor the equations (7.15) do not transform into eachother under (7.11). However, the physical solutions, which are determined by the second factor, do! Summarizing, we conclude that it is obvious that the notion of duality should have an extension that incorporates non-invertible K-matrices. We leave this for future investigation.

7.2. $(\widehat{\mathfrak{sl}_3})_{k,M}$

As argued in refs. 42 and 12, the generalization of the results of the previous section to levels $k>1$ requires the addition of $2(k-1)$ pseudoparticles incorporating the non-abelian statistics of the quasihole operators $\{\phi_\uparrow, \phi_\downarrow\}$. Since these pseudoparticles couple differently to $\{\phi_\uparrow, \phi_\downarrow\}$ than to the composite particle $\phi_{\uparrow\downarrow}=(\phi_\uparrow\phi_\downarrow)$ (i.e., different than the naive coupling given by the composite construction), it appears that the first construction in Section 7.1 does not generalize to higher levels.

It is known that for $(\widehat{\mathfrak{sl}_n})_{k,\,M=0}$ the pseudoparticles couple to the physical particles by means of the matrix $\mathbf{A}_{n-1}^{-1}\otimes\mathbf{A}_k$. Here we have used the result for the restricted Kostka polynomials as given in, e.g., refs. 43, 30, 44, and 45 (see the discussion in ref. 42 for details). Then, by applying the shift map (4.19), we obtain

$$\mathbf{K}'^{M}_{\phi}=\left(\begin{array}{cc:ccc} & & \vdots & & \\ & & \vdots & & \\ \mathbf{A}_2^{-1}\otimes\mathbf{A}_{k-1} & & \vdots & -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ & & \vdots & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \hdashline -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} \\ -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} \\ -\frac{1}{3} & -\frac{2}{3} & \vdots & \frac{(2k-2)M+3}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} & \frac{(4k-4)M+6}{3(2kM+3)} \end{array}\right) \tag{7.16}$$

where the components of $\mathbf{A}_2$ refer to the quasiholes in the **3** and **3***, respectively, and

$$\mathbf{t}'_\phi = \Bigg(\underbrace{0, 0, \ldots, 0}_{2(k-1)} \Bigg| \frac{1}{2kM+3}, \frac{1}{2kM+3}, \frac{2}{2kM+3}\Bigg) \tag{7.17}$$

For instance, for level $k=2$ we have

$$\mathbf{K}'^{M}_\phi = \left(\begin{array}{cc:ccc} \frac{4}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \hdashline -\frac{2}{3} & -\frac{1}{3} & \frac{7M+6}{12M+9} & \frac{7M+6}{12M+9} & \frac{2M+3}{12M+9} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7M+6}{12M+9} & \frac{7M+6}{12M+9} & \frac{2M+3}{12M+9} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2M+3}{12M+9} & \frac{2M+3}{12M+9} & \frac{4M+6}{12M+9} \end{array}\right) \tag{7.18}$$

Note that the matrix $\mathbf{K}'^{M}_\phi$ of (7.16) is not invertible, as was observed for $k=1$ in Section 7.1. Thus, we cannot simply identify the dual sector by performing the transformation (4.15).

To obtain the dual sector we proceed as in Section 7.1. We start with the K-matrix of the principal subspace spanned by $\{J_{-\alpha_1}, J_{-\alpha_2}\}$. As discussed in Appendix B, for $(\widehat{\mathfrak{sl}_3})_k$, this K-matrix is given by $\mathbf{K} = \mathbf{A}_2 \otimes \mathbf{B}_k$ and requires, besides the currents $\{J_{-\alpha_1}, J_{-\alpha_2}\}$ a set of $2(k-1)$ composites

$$(\underbrace{J_{-\alpha_i} \cdots J_{-\alpha_i}}_{l}), \qquad 2 \leqslant l \leqslant k,\ i=1,2 \tag{7.19}$$

Starting with this matrix we introduce additional composites according to the procedure of Section 4.4, beginning with the electron operator $\Psi_\downarrow = (J_{-\alpha_1} J_{-\alpha_2})$ (recall that $\Psi_\uparrow = J_{-\alpha_2}$), and continuing until all composites

$$(\underbrace{\Psi_\uparrow \cdots \Psi_\uparrow}_{n_\uparrow} \underbrace{\Psi_\downarrow \cdots \Psi_\downarrow}_{n_\downarrow}), \qquad n_\uparrow + n_\downarrow \leqslant k \tag{7.20}$$

have been introduced. Note that the set of composites (7.20), for fixed $n_\uparrow + n_\downarrow$, span a $(n_\uparrow + n_\downarrow + 1)$-dimensional irreducible representation of spin $SU(2)$. The electron K-matrix is then the $\frac{1}{2}k(k+3) \times \frac{1}{2}k(k+3)$ submatrix of the resulting $\mathbf{K}$ obtained by omitting the composites which cannot be written in terms of electron operators only. Let us be illustrate this procedure the case of $k = 2$. Starting with the principal subspace K-matrix

$$\mathbf{K} = \begin{pmatrix} 2 & -1 & \vdots & 2 & -1 \\ -1 & 2 & \vdots & -1 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & -1 & \vdots & 4 & -2 \\ -1 & 2 & \vdots & -2 & 4 \end{pmatrix} \tag{7.21}$$

we introduce, respectively, the composites $\Psi_\downarrow = (J_{-\alpha_1} J_{-\alpha_2})$, $(J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_1}))$, $(J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_2}))$, and $(J_{-\alpha_2}((J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_1})))$. Then, after removing the rows and columns corresponding to $J_{-\alpha_1}$, $(J_{-\alpha_1} J_{-\alpha_1})$ and $(J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_1}))$, we obtain

$$\mathbf{K}'_e = \begin{pmatrix} 2 & 1 & \vdots & 2 & 2 & 1 \\ 1 & 2 & \vdots & 1 & 2 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 1 & \vdots & 4 & 3 & 2 \\ 2 & 2 & \vdots & 3 & 4 & 3 \\ 1 & 2 & \vdots & 2 & 3 & 4 \end{pmatrix}, \qquad \mathbf{t}'_e = -\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{s}'_e = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} \tag{7.22}$$

Similarly, one obtains the electron K-matrix for $(\widehat{\mathfrak{sl}_3})_{k,\,M=0}$ at higher levels, and the generalization to arbitrary M follows, as before, by applying the shift map (4.18). Unfortunately, the procedure described above is ambiguous. The resulting K-matrix depends on the order in which the composites are taken as well as the precise identification of the clusters (7.20) with the original clusters (7.19), e.g., should we identify $(\Psi_\downarrow \Psi_\downarrow)$ with $(J_{-\alpha_1}(J_{-\alpha_1}(J_{-\alpha_2} J_{-\alpha_2})))$ or $((J_{-\alpha_1} J_{-\alpha_1})(J_{-\alpha_2} J_{-\alpha_2}))$? Ultimately, the "correct" matrix $\mathbf{K}_e$ is selected by the requirement that the complete spectrum can be built out of the quasihole and electron operators or, more concretely, that the characters of $(\widehat{\mathfrak{sl}_3})_{k,M}$ can be written as a linear combination of UCPFs based on $\mathbf{K}_\phi \oplus \mathbf{K}_e$. A nontrivial (and highly selective) check is whether the central charge, given by (4.7), works out correctly, i.e., whether $c_\phi + c_e = 8k/(k+3)$, for the K-matrices (7.16) and the "appropriate" generalization of (7.22) to higher levels and arbitrary M. We have checked this numerically for low values of k and M as well as exactly, for all k, in the $M \to \infty$

limit, in which case the central charge is entirely concentrated in the ϕ-sector. We refrain from giving the explicit matrices $\mathbf{K}_e$ until we have performed an additional simplifying reduction.

First observe that, for $k=2$, the matrix $\mathbf{K}'_e$ of Eq. (7.22) is invertible, in contrast to the matrix $\mathbf{K}'^{M}_{\phi}$ of (7.18). One could therefore simply have started with $\mathbf{K}'_e$ and have obtained the dual sector by the duality transformations (4.15). This would result in a ϕ-sector, different from the one discussed above, with two physical quasiholes and three pseudoparticles. Unfortunately, this procedure breaks down, in general, for higher k as the matrices $\mathbf{K}_e$, constructed according to the procedure outlined above, are no longer invertible. However, note that the matrix (7.22) can be reduced to an equivalent 4×4 matrix by inverting the composite procedure—in this case by removing $(\Psi_\uparrow \Psi_\downarrow)$ in the fourth column, since this column can be created by applying $\mathscr{C}_{12}$. This procedure works for general $k>1$ and leads to a $2k\times 2k$ electron K-matrix, for the composites (7.19) with either $n_\downarrow=0$ or $n_\uparrow=0$ (i.e., we lose the $SU(2)$ multiplet structure), given by

$$\mathbf{K}_e=\begin{pmatrix} 2 & 0 & 2 & 0 & \cdots & 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 & \cdots & 0 & 2 & 1 & 2 \\ 2 & 0 & 4 & 0 & & 4 & 1 & 4 & 2 \\ 0 & 2 & 0 & 4 & & 1 & 4 & 2 & 4 \\ \vdots & \vdots & & & & & & \vdots & \vdots \\ 2 & 0 & 4 & 1 & & 2(k-1) & k-2 & 2(k-1) & k-1 \\ 0 & 2 & 1 & 4 & & k-2 & 2(k-1) & k-1 & 2(k-1) \\ 2 & 1 & 4 & 2 & \cdots & 2(k-1) & k-1 & 2k & k \\ 1 & 2 & 2 & 4 & \cdots & k-1 & 2(k-1) & k & 2k \end{pmatrix} \tag{7.23}$$

and

$$\begin{aligned} \mathbf{t}_e &= -(1,\, 1;\, 2,\, 2; \ldots;\, k,\, k) \\ \mathbf{s}_e &= (1,\, -1;\, 2,\, -2; \ldots;\, k,\, -k) \end{aligned} \tag{7.24}$$

The generalization $\mathbf{K}^M_e$ to arbitrary M follows by applying the shift map, in this case by adding the matrix $M(\mathbb{J}_2 \otimes \mathbf{D})$ where $\mathbb{J}_2$ is the 2×2 matrix with all entries equal to 1, and $(\mathbf{D})_{ij}=ij$ $(1\leqslant i,\, j\leqslant k)$ (see ref. 14 for an explicit expression in the case $k=2$). This matrix is invertible, so we simply define $\mathbf{K}^M_\phi=(\mathbf{K}^M_e)^{-1}$. A convenient permutation of rows and columns of $\mathbf{K}^M_\phi$ leads to the following matrix

$$
(\mathbf{K}_\phi^M)^{\text{perm}} = \left(\begin{array}{cccccc:cc}
 & & & & & & 0 & -\frac{1}{3} \\
 & & & & & & 0 & -\frac{2}{3} \\
 & & \mathbf{A}_2^{-1} \otimes \mathbf{A}_{k-1} & & & & & \\
 & & & & & & -\frac{2}{3} & 0 \\
 & & & & & & -\frac{1}{3} & 0 \\
\hdashline
0 & 0 & & & -\frac{2}{3} & -\frac{1}{3} & \frac{(4k-1)\,M+6}{3(2kM+3)} & \frac{-M}{3(2kM+3)} \\
-\frac{1}{3} & -\frac{2}{3} & & & 0 & 0 & \frac{-M}{3(2kM+3)} & \frac{(4k-1)\,M+6}{3(2kM+3)}
\end{array}\right) \tag{7.25}
$$

containing two physical particles and $2(k-1)$ pseudoparticles. Also,

$$
\begin{aligned}
\mathbf{t}_\phi &= \left(0,\, 0;\, 0,\, 0; \ldots;\, \frac{1}{2kM+3},\, \frac{1}{2kM+3}\right) \\
\mathbf{s}_\phi &= (0,\, 0;\, 0,\, 0; \ldots;\, -1,\, 1)
\end{aligned} \tag{7.26}
$$

as one would expect. We have checked that the total central charge $c_e + c_\phi$ for Eqs. (7.23) and (7.25) works out correctly, namely $c_e + c_\phi = 8k/(k+3)$. Moreover, we have checked for low values of k that the equation for λ_{tot}, resulting from the IOW equations based on (7.25), are identical to those based on (7.16). Furthermore, in all formulations, the equations (4.10) and (4.11) are consistent with (7.1).

For $k = 2, 3$, we checked the small x behaviour for λ_{tot}, Eq. (4.5). We again expect the constants α to be the quantum dimensions of the associated conformal field theory. Using some results in ref. 23, these quantum dimensions are given by

$$
\alpha_k = 1 + 2\cos\left(\frac{2\pi}{k+3}\right) \tag{7.27}
$$

For $k=2$, the equation for λ_{tot} reads (upon taking $x_\uparrow = x_\downarrow = x$)

$$(\lambda_{\text{tot}}^{1/2}-1)^2 = x^2\lambda_{\text{tot}}^{(8M+5)/(8M+6)} + x\lambda_{\text{tot}}^{(6M+4)/(8M+6)} - x\lambda_{\text{tot}}^{(2M+1)/(8M+6)} \qquad (7.28)$$

which leads to the following small x behaviour

$$\lambda_{\text{tot}} = 1 + 2\left(\frac{1+\sqrt{5}}{2}\right) x + O(x^2) \qquad (7.29)$$

in agreement with $\alpha_2 = (1+\sqrt{5})/2$ from (7.27); the extra factor 2 comes from the sum over the two physical particles, see Eq. (4.5). For $k=3$ we find

$$(\lambda_{\text{tot}}^{1/2}-1) = x\lambda_{\text{tot}}^{(8M+3)/(6(6M+3))}(\lambda_{\text{tot}}^{1/6}+1)^{1/3}\,(\lambda_{\text{tot}}^{1/3}+1)^{2/3} \qquad (7.30)$$

which gives $\alpha_3 = 2$, consistent with (7.27). Note that for the abelian case $k=1$, we find for the small $x_{\uparrow,\downarrow}$-behaviour, using (7.5),

$$\lambda_{\text{tot}} = 1 + (x_\uparrow + x_\downarrow) + O(x^2) \qquad (7.31)$$

in agreement with (7.27) and the fact that for $k=1$ we have an abelian state.

As was the case for the spin polarized states of Section 6, also for the non-abelian spin singlet states a particle behaving as a supercurrent can be identified in the non-magnetic limit. The situation here is slightly more complicated than in the case of the spin polarized states discussed in Section 6. This is because in the formulation above, there is no candidate particle with the property that all the statistics parameters go to zero in the limit $q \to 0$ (with $q = 1/\nu = M + 3/2k$). However, if one acts with $\mathscr{C}_{2k-1,2k}$ on $\mathscr{S}_M \mathbf{K}_e$, with $\mathbf{K}_e$ given by Eq. (7.23), one introduces a composite with charge $-2k$ and spin 0, which has the desired properties. In the ϕ-sector, the particle content is changed to one quasihole and $2k$ pseudoparticles, of which a few carry spin.

The possibility to introduce a composite with the right properties enables one to repeat the discussion of Section 6, with the only difference that the flux quantum in this case equals $h/2ke$. So, also in this case, we can identify a supercurrent in the non-magnetic limit.

8. DISCUSSION

In this paper we derived the K-matrix structure for two classes of so called non-abelian quantum Hall states, putting the results of ref. 14 on a firmer basis. In doing so, we extensively made use of a duality between the edge electron and quasihole excitations. The abelian formalism was extended to include electron spin, in order to be able to treat spin singlet states. Moreover, we showed that many results of the abelian K-matrix formulation for hierarchy states also hold for our generalized K-matrices, thereby justifying their name. We would like to stress that the non-abelian states of refs. 24 and 10 are not hierarchical states; the K-matrix structure is necessary as a bookkeeping device for the non-abelian statistics.

An important concept we did not discuss is the torus degeneracy;(46) it is not clear at the moment how to generalize this to the non-abelian case (some remarks are made in Appendix D). Another important issue to be settled has to do with the cases where the pseudoparticles do carry spin (or charge). These may arise by creating extra composites in the electron sector; by the duality, the ϕ sector changes accordingly, and pseudoparticles carrying spin may arise. The formulas Eq. (2.2) then need a proper adjustment, because they do not give the same result any more, and the physical quantities like the filling factors need to be invariant under the introduction of extra composites. We would like to remark that a description in which the pseudoparticles do not carry spin or charge is possible in the cases we examined, and the various physical quantities were obtained correctly.

As for the Laughlin wave functions, one would like to have a Landau–Ginzburg field theory describing the excitations for the non-abelian states. The backbone of such a theory will be a Chern–Simons term, in which the gauge fields are coupled in a special way. We expect that the K-matrices derived in this paper will play a crucial role. From a Landau–Ginzburg theory (using the K-matrices etc. from the electronic sector), one should be able to identify the possible excitations in the ϕ-sector, as vortex solutions of the classical equations of motion. Identifying this Landau–Ginzburg theory is left for future investigations (see ref. 47 for related studies).

Another interesting issue for the non-abelian states is the determination of the degeneracies of the states when extra flux is applied through the sample. These degeneracies can be calculated using conformal field theory techniques, and can, interestingly, be simulated on a computer using a special, ultra local, interaction for the electron interaction. For the Pfaffian, exact counting results were obtained in ref. 48; the more general Read–Rezayi states were treated in ref. 49. Counting results for the NASS states will be given elsewhere.(26)

Finally, while our discussion of fqH-bases of conformal field theories based on quasiparticles with a statistics matrix $\mathbf{K} \oplus \mathbf{K}^{-1}$ was restricted to $(\widehat{\mathfrak{sl}_n})_k$ (for $n = 2, 3$), it is obvious that such a description generalizes to more general conformal field theories (see ref. 31 for more examples), even though these may not have an interpretation in the context of the fractional quantum Hall effect.

APPENDIX A. BASIC HYPERGEOMETRIC SERIES

Consider the basic hypergeometric series

$$
\begin{aligned}
&{}_r\phi_s(a_1,..., a_r; b_1,..., b_s; q, z)\\
&\quad =\sum_{m\geqslant 0} \frac{(a_1;q)_m\,(a_2;q)_m\cdots(a_r;q)_m}{(q;q)_m\,(b_1;q)_m\cdots(b_s;q)_m}\,((-1)^m\,q^{1/2\,m(m-1)})^{1+s-r}\,z^m \qquad \text{(A.1)}
\end{aligned}
$$

where

$$
(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k) \qquad \text{(A.2)}
$$

We have the q-Pfaff–Saalschütz sum[50, 51]

$$
{}_3\phi_2(a, b, q^{-n}; c, abq^{1-n}/c; q, q)=\frac{(c/a;q)_n\,(c/b;q)_n}{(c;q)_n\,(c/ab;q)_n} \qquad \text{(A.3)}
$$

Taking $b=0$ in (A.3) gives the q-Chu–Vandermonde sum

$$
{}_2\phi_1(a, q^{-n}; c; q, q)=\frac{(c/a;q)_n}{(c;q)_n}\,a^n \qquad \text{(A.4)}
$$

Now, taking $a=q^{-m_1}$, $b=q^{M_1+M_2-(m_1+m_1)+1}$, $c=q^{M_2-(m_1+m_2)+1}$ and $n=m_2$ in (A.3) gives

$$
\begin{aligned}
\begin{bmatrix} M_1\\ m_1\end{bmatrix}\begin{bmatrix} M_2\\ m_2\end{bmatrix}&=\sum_{m\geqslant 0} q^{(m_1-m)(m_2-m)}\begin{bmatrix} M_1-m_2\\ m_1-m\end{bmatrix}\\
&\quad\times\begin{bmatrix} M_2-m_1\\ m_2-m\end{bmatrix}\begin{bmatrix} M_1+M_2-(m_1+m_2)+m\\ m\end{bmatrix} \qquad \text{(A.5)}
\end{aligned}
$$

Taking $a=q^{-m_1}$, $c=q^{M_2-(m_1+m_2)+1}$ and $n=m_2$ in (A.4) gives

$$
\frac{1}{(q)_{m_1}}\begin{bmatrix} M_2\\ m_2\end{bmatrix}=\sum_{m\geqslant 0} q^{(m_1-m)(m_2-m)}\,\frac{1}{(q)_m\,(q)_{m_1-m}}\begin{bmatrix} M_2-m_1\\ m_2-m\end{bmatrix} \qquad \text{(A.6)}
$$

while taking $a=q^{-m_1}$, $n=m_2$ and $c=0$ in (A.4) gives

$$
\frac{1}{(q)_{m_1}\,(q)_{m_2}}=\sum_{m\geqslant 0} q^{(m_1-m)(m_2-m)}\,\frac{1}{(q)_m\,(q)_{m_1-m}\,(q)_{m_2-m}} \qquad \text{(A.7)}
$$

APPENDIX B. THE PRINCIPAL SUBSPACE

In this appendix we review an important result of refs. 38 and 39 which is used throughout the paper. Consider an affine Lie algebra $\hat{\mathfrak{g}}_k$ (see, e.g., ref. 52 for notation and definitions). If $L(\Lambda)$ is the integrable highest weight module of $\hat{\mathfrak{g}}_k$ with highest weight Λ and highest weight vector v_Λ, then the principal subspace $W(\Lambda) \subset L(\Lambda)$ is defined to be the subspace generated from v_Λ by the negative modes of the positive simple root currents $J_{\alpha_i}(z)$.

The character of the principal subspace $W(\Lambda)$ of the integrable highest weight module $L(\Lambda)$ for $\Lambda = k_0 \Lambda_0 + k_j \Lambda_j$ $(1 \leqslant j \leqslant n,\ k_0 + k_j = k)$ of $(\widehat{\mathfrak{sl}_{n+1}})_k$ was determined in refs. 38 and 39.[6] It is given by the UCPF

$$\mathrm{ch}_W = \sum_{\mathbf{p}} \left(\prod z_i^{s p_i^{(s)}} \right) \frac{q^{\frac{1}{2} \mathbf{p} \cdot \mathbf{K} \cdot \mathbf{p} + \mathbf{Q}_j \cdot \mathbf{p}}}{\prod_i \prod_s (q)_{p_i^{(s)}}} \tag{B.1}$$

where

$$\mathbf{K} = \mathbf{A}_n \otimes \mathbf{B}_k, \qquad \mathbf{Q}_j = \mathbf{e}_j \otimes (\underbrace{0, \ldots, 0}_{k_0}, 1, 2, \ldots, k_j) \tag{B.2}$$

and z_i denotes the (generalized) fugacity of the current J_{α_i}. Also, $(\mathbf{A}_n)_{ij} = 2\delta_{ij} - \delta_{i-1,j} - \delta_{i+1,j}$ is the Cartan matrix of $\mathfrak{sl}_{n+1}$, i.e.,

$$\mathbf{A}_n = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{B.3}$$

and $(\mathbf{B}_k)_{rs} = \min(r, s)|_{r,s=1,\ldots,k}$, i.e.,

$$\mathbf{B}_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & k \end{pmatrix} \tag{B.4}$$

[6] The character of the principal subspace $W(\Lambda)$ for more general level k modules $L(\Lambda)$ is apparently not yet known.

Furthermore, in (B.1), we have written $\mathbf{p}=(p_j^{(s)})_{j=1,\dots,n}^{s=1,\dots,k}$ with respect to $(\mathbf{A}_n)_{ij}\otimes(\mathbf{B}_k)_{rs}$.

APPENDIX C. $(\widehat{\mathfrak{sl}_2})_{k,M}$ CHARACTER

The UCPF character for $(\widehat{\mathfrak{sl}_2})_{k=1,M}$ was discussed in ref. 6 (see also ref. 28). Here we discuss the q-Pfaffian case, i.e., $k=2$. For convenience we put $q=M+1$.

C.1. Quasihole Sector

In ref. 9, finitized partition sums $X_L=X_{l=(8L-q-2)/16q}$ and $Y_L=Y_{l=(8L+q-6)/16q}$ for the quasihole sector of the q-pfaffian CFT were introduced. X_L (Y_L) are restricted by requiring that the total charge be an even (odd) multiple of $1/2q$. In ref. 9, it was established that the following recursion relations hold

$$\begin{aligned} X_L &= X_{L-2q}+xq^{(8L-q-2)/16q}(Y_L+Y_{L-q}) \\ Y_L &= Y_{L-2q}+xq^{(8L+q-6)/16q}X_{L-1} \end{aligned} \tag{C.1}$$

or, equivalently,

$$X_L=X_{L-2q}+q^{1/2}(X_{L-q}-X_{L-3q})+x^2q^{(2L-1)/2q}X_{L-1} \tag{C.2}$$

By putting $X_L/X_{L-q}\sim\lambda_{\text{tot}}^{1/2q}$, for large L, we reproduce Eq. (6.4). To build the entire spectrum of the $(\widehat{\mathfrak{sl}_2})_{k=2,M}$ conformal field theory we need $3q$ sectors whose initial conditions are given in Table I. The vacua of the sectors are labeled by, respectively, charge and the $\mathfrak{sl}_2$ irrep in which they appear for $M=0$ (the labels $\mathbb{1}$, σ and ψ stand for the $\mathfrak{sl}_2$ singlet, doublet

Table I. $(\widehat{\mathfrak{sl}_2})_{k=2,M}$: Qausihole Sector

Sector	Initial conditions	$\mathbf{Q}_\phi$
$\left\|-\frac{re}{q},\mathbb{1}\right\rangle$	$X_s=1,\ Y_s=0$	$\left(0,\frac{s}{2q}\right)$
$\left\|-\frac{(2r+1)e}{2q},\sigma\right\rangle$	$X_{2q-r}=xq^{(15q-2-8r)/16q},\ Y_{2q-r}=1$	$\left(-\frac{1}{2},\frac{5q-1-2r}{4q}\right)$
$\left\|-\frac{re}{q},\psi\right\rangle$	$X_s=1,\ Y_s=0$	$\left(0,\frac{s}{2q}\right)$

and triplet, respectively, in analogy with the Ising model). The parameter r takes the values $r=0, 1,..., q-1$.

The solutions to (C.1) can be written in terms of finitized UCPFs with (cf. (6.3))

$$\mathbf{K}_\phi = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{q+1}{4q} \end{pmatrix} \tag{C.3}$$

Indeed, the recursion relations (5.12), with $\mathbf{K}=\mathbf{K}_\phi$ and $\mathbf{Q}+\mathbf{u}=0$, are explicitly given by

$$\begin{aligned} P_{L_1, L_2} &= P_{L_1-1, L_2} + q^{L_1-1/2} P_{L_1-1, L_2+1/2} \\ P_{L_1, L_2} &= P_{L_1, L_2-1} + xq^{L_1-(q+1)/8q} P_{L_1+1/2, L_2-(q+1)/4q} \end{aligned} \tag{C.4}$$

and lead to (C.1) upon identifying

$$X_L = q^{1/4\, Q_1^2} P_{0,\, L/2q}, \qquad Y_L = q^{\frac{1}{4} Q_1^2 - \frac{1}{16}} P_{-1/2,\, L/2q+(q-1)/4q} \tag{C.5}$$

The values for $\mathbf{Q}_\phi$ in each sector are listed in Table I, while the parameters $s=0,..., 2q-1$, in Table I, are given in Table II.

C.2. Electron Sector

For the electron sector of the q-pfaffian, the paper[9] introduced the truncated partition sums Ω_L, which contain all states constructed from the edge electron operator Ψ_{-s} with $s \leqslant L-(q-1)/2$. It satisfies the recursion relation

$$\begin{aligned} \Omega_L = \Omega_{L-1} &+ yq^{L-1/2(q-1)} \Omega_{L-q} + y^2 q^{2L-(2q-1)} \Omega_{L-2q} \\ &- y^3 q^{3L-1/2(9q-5)} \Omega_{L-3q} \end{aligned} \tag{C.6}$$

and results in Eq. (6.7) by putting $\Omega_L/\Omega_{L-1} \sim \mu_{\text{tot}}$ for large L. In this case the recursion relation does not appear to be solved by finitized UCPFs.

Table II. Relation Between Sectors and Parameter *s*

Sector	$\|0, \mathbb{1}\rangle$	$\|-\frac{e}{q}, \psi\rangle$	$\|-\frac{2e}{q}, \mathbb{1}\rangle$	$\cdots$	$\|0, \psi\rangle$	$\|-\frac{e}{q}, \mathbb{1}\rangle$	$\|-\frac{2e}{q}, \psi\rangle$	$\cdots$
s	0	1	2	$\cdots$	q	$q+1$	$q+2$	$\cdots$ $2q-1$

Table III. $(\widehat{\mathfrak{sl}_2})_{k=2,\,M}$**: Electron Sector**

Sector	Initial conditions	$\mathbf{Q}_\phi$
$\left\vert -\frac{re}{q}, \mathbb{1}\right\rangle$	$\Omega_{r-1} = \cdots = \Omega_{q+r-1} = 1$	$(r, 2r)$
$\left\vert -\frac{(2r+1)}{2q}, \sigma\right\rangle$	$\Omega_r = \cdots = \Omega_{q+r-1} = 1$	$(r, 2r+1)$
$\left\vert -\frac{re}{q}, \psi\right\rangle$	$\Omega_{-q+r-1} = q^{1/2(q-1)-r}/y,\ \Omega_r = 1$	$(r-1, 2r+1;\ -1,\ -1,\ -1,\dots)$

However, there exists a recursion relation, leading to the same equation for μ_{tot}, that is solved by a finitized UCPF and differs from the solution to (C.6) by terms of order q^L (i.e., belongs to the same universality class, see the discussion in Section 5.3) and thus gives the correct solution in the limit $L \to \infty$. The UCPF is based on the K-matrix (cf. (6.6))

$$\mathbf{K}_e = \begin{pmatrix} q+1 & 2q \\ 2q & 4q \end{pmatrix} \tag{C.7}$$

The initial conditions and values for $\mathbf{Q}_e$ in each sector are listed in Table III.

There is a slight subtlety in the case of the sectors $|-re/q, \psi\rangle$. These vectors do not correspond to an extremal vector in the $(\widehat{\mathfrak{sl}_2})_{k=2,\,M}$ modules. Thus the results of Appendix B do not apply. While the exclusion statistics of the currents is unchanged, and hence the K-matrix is still given by (C.7), it can easily be shown that the extremal vectors in the modules cannot be reproduced by any two dimensional vector $\mathbf{Q}$. In fact, to correctly reproduce the extremal vectors one needs an infinite dimensional vector $\mathbf{Q}$ (given in Table III) with a corresponding infinite dimensional K-matrix $\mathbf{K}_e^{(\infty)}$ that is equivalent to (C.7) by the composite construction. Specifically, one introduces derived matrices $\mathbf{K}_e^{(n)}$ and associated generalized fugacities $\mathbf{z}^{(n)}$ by

$$\mathbf{K}_e^{(1)} = \mathscr{C}_{12}\mathbf{K}_e = \begin{pmatrix} q+1 & 2q+1 & 3q+1 \\ 2q+1 & 4q & 6q \\ 3q+1 & 6q & 9q+1 \end{pmatrix}, \qquad \mathbf{z}^{(1)} = \begin{pmatrix} z \\ z^2 \\ z^3 \end{pmatrix} \tag{C.8}$$

$$\mathbf{K}_e^{(2)} = \mathscr{C}_{23}\mathbf{K}_e^{(1)} = \begin{pmatrix} q+1 & 2q+1 & 3q+1 & 5q+2 \\ 2q+1 & 4q & 6q+1 & 10q \\ 3q+1 & 6q+1 & 9q+1 & 15q+1 \\ 5q+2 & 10q & 15q+1 & 25q+1 \end{pmatrix}, \qquad \mathbf{z}^{(2)} = \begin{pmatrix} z \\ z^2 \\ z^3 \\ z^5 \end{pmatrix} \tag{C.9}$$

and, ultimately,

$$\mathbf{K}_e^{(\infty)} = \lim_{n\to\infty} \mathbf{K}_e^{(n)} = \lim_{n\to\infty} \mathscr{C}_{2,n}\mathscr{C}_{2,n-1}\cdots\mathscr{C}_{23}\mathscr{C}_{12}\mathbf{K}_e$$

$$= \begin{pmatrix} q+1 & 2q+1 & 3q+1 & 5q+2 & 7q+3 & \cdots & (2k+1)\,q+k & \cdots \\ 2q+1 & 4q & 6q+1 & 10q+1 & 14q+1 & \cdots & 2(2k+1)\,q+1 & \cdots \\ 3q+1 & 6q+1 & 9q+1 & 15q+1 & 21q+2 & \cdots & 3(2k+1)\,q+(k-1) & \\ 5q+2 & 10q+1 & 15q+1 & 25q+1 & 35q+1 & \cdots & 5(2k+1)\,q+(k-2) & \\ 7q+3 & 14q+1 & 21q+2 & 35q+1 & 49q+1 & \cdots & 7(2k+1)\,q+(k-3) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ & & & & & & (2k+1)^2\,q+1 & \\ & & & & & & & \ddots \end{pmatrix} \tag{C.10}$$

while

$$\mathbf{z}^{(\infty)} = (z, z^2; z^3, z^5, z^7,...) \tag{C.11}$$

For every finite n, the UCPF based on $(\mathbf{K}_e^{(n)}; \mathbf{Q}_e^{(n)})$, where $\mathbf{Q}_e^{(n)}$ is the $(n+2)$-dimensional truncation of the vector $\mathbf{Q}_e$ in Table III, gives an accurate description of the module up to some level (which appears to be at at least polynomially increasing with n). To describe the entire module accurately, one needs to take the limit $n \to \infty$.

C.3. The Character

Combining the $3q$ sectors in Tables I and II should reproduce the spectrum of the chiral $(\widehat{\mathfrak{sl}_2})_{2,M}$ conformal field theory. Consider the combination of UCPFs

$$Z_{\text{tot}} = \sum_{k=1}^{3q} a_{(k)} Z_\infty(\mathbf{K}_e; \mathbf{Q}_e^{(k)})\, Z_{\infty/2}(\mathbf{K}_\phi; \mathbf{Q}_\phi^{(k)}, \mathbf{u}_\phi^{(k)}) \tag{C.12}$$

where the coefficients $a_{(k)}$ and vectors $\mathbf{Q}_e^{(k)}$, $\mathbf{Q}_\phi^{(k)} = -\mathbf{u}_\phi^{(k)}$ are given in Table IV and where

$$Z_{\infty/2}(\mathbf{K}_\phi; \mathbf{Q}_\phi^{(k)}, \mathbf{u}_\phi^{(k)}) \equiv q^{1/4(Q_1^{(k)})^2} \left(Z(\mathbf{K}_\phi; \mathbf{Q}_\phi^{(k)}, \mathbf{u}_\phi^{(k)}) + q^{-1/16} Z\left(\mathbf{K}_\phi; \mathbf{Q}_\phi^{(k)}, \mathbf{u}_\phi^{(k)} - \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) \right) \tag{C.13}$$

Table IV. $(\widehat{\mathfrak{sl}_2})_{k=2,M}$: **Character**

Sector	$\mathbf{Q}_\phi$	$\mathbf{Q}_e$	$a_{(k)}$
$\left\vert -\frac{re}{q}, \mathbb{1}\right\rangle$	$\left(0, \frac{s}{2q}\right)$	$(r, 2r)$	$x^{-2r}q^{r^2/2q}$
$\left\vert -\frac{(2r+1)e}{2q}, \sigma\right\rangle$	$\left(-\frac{1}{2}, \frac{5q-1-2r}{4q}\right)$	$(r, 2r+1)$	$x^{-(2r+1)}q^{((2r+1)^2/8q)+1/16}$
$\left\vert -\frac{re}{q}, \psi\right\rangle$	$\left(0, \frac{s}{2q}\right)$	$(r-1, 2r+1; -1; -1;...)$	$x^{-2r}q^{(r^2/2q)+1/2}$

corresponds to the limit

$$\lim_{L\to\infty} \sum_{t=0}^{2q-1} (X_{L-t} + Y_{L-t}) \tag{C.14}$$

We have numerically checked that (C.12) indeed equals the $(\widehat{\mathfrak{sl}_2})_{k=2,M}$ character

$$Z_{\text{tot}} = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} \left(x^{2n} q^{(1/2q)\, n^2} \prod_{k\geqslant 1} (1+q^{k-1/2}) + x^{2n+1} q^{(1/2q)(n+1/2)^2+1/16} \prod_{k\geqslant 1} (1+q^k) \right) \tag{C.15}$$

corresponding to a free fermion and a boson compactified on a circle of radius $R^2 = q$. It should be possible to prove the equality of (C.12) and (C.15) along the lines of ref. 31 (see also Appendix D). Finally, we note that the number of summands in (C.12) equals the torus degeneracy for the q-Pfaffian computed in ref. 25.

APPENDIX D. $(\widehat{\mathfrak{sl}_3})_{k,M}$ CHARACTER

We will restrict the discussion in this section to $(\widehat{\mathfrak{sl}_3})_{k,M}$ for level $k=1$.

D.1. Quasihole Sector

The recursion relation for the quasiholes $(\phi_\uparrow, \phi_\downarrow)$ in $(\widehat{\mathfrak{sl}_3})_{k,M}$ for $k=1$ and $M=0$ was worked out in refs. 20 and 13. The generalization to arbitrary M reads

$$X_L = X_{L-(2M+3)} + (x_\uparrow + x_\downarrow)\, q^{(2L-(M+2))/(2(2M+3))} X_{L-(M+2)} + x_\uparrow x_\downarrow q^{(2L-1)/(2M+3)} X_{L-1} \tag{D.1}$$

By putting $X_L/X_{L-1} \sim \lambda_{\text{tot}}^{1/(2M+3)}$ we recover the IOW-equation (7.5). To build the entire spectrum out of quasiholes and electrons we need $3M+4$ sectors whose initial conditions are given in Table V. The vacua of the sectors are labeled by, respectively, charge, spin, and the $\mathfrak{sl}_3$ irrep in which they occur for $M=0$. The parameter r takes the values $r = 1, 2, \ldots, M+1$.

The solution to (D.1) can be written in terms of finitized UCPFs (see (5.10)). Indeed, the recursion relations (5.12) with (see (7.3))

$$\mathbf{K}_\phi = \frac{1}{2M+3}\begin{pmatrix} M+2 & -(M+1) \\ -(M+1) & M+2 \end{pmatrix} \tag{D.2}$$

and $\mathbf{Q}+\mathbf{u} = (0, 0)$ are explicitly given by

$$\begin{aligned} P_{L_1, L_2} &= P_{L_1-1, L_2} + x_\uparrow q^{L_1-(M+2)/(2(2M+3))} \\ &\quad \times P_{L_1-(M+2)/(2M+3),\, L_2+(M+1)/(2M+3)} \\ P_{L_1, L_2} &= P_{L_1, L_2-1} + x_\downarrow q^{L_2-(M+2)/(2(2M+3))} \\ &\quad \times P_{L_1+(M+1)/(2M+3),\, L_2-(M+2)/(2M+3)} \end{aligned} \tag{D.3}$$

Setting $X_L \equiv P_{L/(2M+3),\, L/(2M+3)}$ leads to (D.1). The values for $\mathbf{Q} = -\mathbf{u}$ in the various sectors, as determined by the initial conditions, are given in Table V.

Table V. $(\widehat{\mathfrak{sl}_3})_{k=1,M}$: Quasihole Sector

Sector	Initial conditions	$\mathbf{Q}_\phi$
$\vert 0, -, \mathbb{1}\rangle$	$X_0 = 1$	$(0, 0)$
$\left\vert -\frac{2re}{3}, -, \mathbf{3}\right\rangle$	$X_{2M+3-r} = 1$	$\left(\frac{2M+3-r}{2M+3}, \frac{2M+3-r}{2M+3}\right)$
$\left\vert -\frac{(2r-1)\,e}{3}, \uparrow, \mathbf{3}^*\right\rangle$	$X_{M+2-r} = 1$	$\left(\frac{M+2-r}{2M+3}, \frac{M+2-r}{2M+3}\right)$
$\left\vert -\frac{(2r-1)\,e}{3}, \downarrow, \mathbf{3}^*\right\rangle$	$X_{M+2-r} = 1$	$\left(\frac{M+2-r}{2M+3}, \frac{M+2-r}{2M+3}\right)$

D.2. Electron Sector

The recursion relations for the electrons $(\Psi_\uparrow, \Psi_\downarrow)$ are given by

$$\Omega_L = \Omega_{L-1} + (y_\uparrow + y_\downarrow)\, q^{L-M/2} \Omega_{L-(M+2)} + y_\uparrow y_\downarrow q^{2L-(2M+1)} \Omega_{L-(2M+3)} \tag{D.4}$$

with initial conditions listed in Table VI. They can be solved by finitized UCPFs with (see (7.9))

$$\mathbf{K}_e = \begin{pmatrix} M+2 & M+1 \\ M+1 & M+2 \end{pmatrix} \tag{D.5}$$

and $\mathbf{Q} + \mathbf{u} = (1, 1)$, by putting $\Omega_L = P_{L,L}$. The values for $\mathbf{Q}_e$ in the various sectors are listed in Table VI.

D.3. Character

Combining the $3M+4$ sectors in Tables V and VI should reproduce the spectrum of the chiral $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ conformal field theory. Indeed, consider the following combination of UCPFs

$$Z_{\text{tot}} = \sum_{k=0}^{3M+3} a_{(k)} Z_\infty(\mathbf{K}_e; \mathbf{Q}_e^{(k)})\, Z_\infty(\mathbf{K}_\phi; \mathbf{Q}_\phi^{(k)}) \tag{D.6}$$

where the coefficients $a_{(k)}$ are defined in Table VII.

Table VI. $(\widehat{\mathfrak{sl}_3})_{k=1,M}$: Electron Sector

Sector	Initial conditions	$\mathbf{Q}_\phi$
$\lvert 0, -, \mathbb{1}\rangle$	$\Omega_{-1} = \Omega_0 = \cdots = \Omega_M = 1$	(0, 0)
$\left\lvert -\frac{2re}{3}, -, \mathbf{3}\right\rangle$	$\Omega_{r-1} = \Omega_r = \cdots = \Omega_{M+r} = 1$	(r, r)
$\left\lvert -\frac{(2r-1)e}{3}, \uparrow, \mathbf{3}^*\right\rangle$	$\Omega_{r-1} = \Omega_r = \cdots = \Omega_{M+r-1} = 1$ $\Omega_{M+r} = 1 + y_\uparrow q^{1/2(M+2)+(r-1)}$	$(r-1, r)$
$\left\lvert -\frac{(2r-1)e}{3}, \downarrow, \mathbf{3}^*\right\rangle$	$\Omega_{r-1} = \Omega_r = \cdots = \Omega_{M+r} = 1$	(r, r)

Table VII. $(\widehat{\mathfrak{sl}_3})_{k=1,M}$**: Character**

Sector	$\mathbf{Q}_\phi$	$\mathbf{Q}_e$	$a_{(k)}$
$\vert 0, -, \mathbb{1}\rangle$	$(0, 0)$	$(0, 0)$	1
$\left\vert -\frac{2re}{3}, -, \mathbf{3}\right\rangle$	$\left(\frac{2M+3-r}{2M+3}, \frac{2M+3-r}{2M+3}\right)$	(r, r)	$(x_\uparrow x_\downarrow)^{-r}\, q^{r^2/(2M+3)}$
$\left\vert -\frac{(2r-1)\,e}{3}, \uparrow, \mathbf{3}^*\right\rangle$	$\left(\frac{M+2-r}{2M+3}, \frac{M+2-r}{2M+3}\right)$	$(r-1, r)$	$x_\uparrow (x_\uparrow x_\downarrow)^{-r}\, q^{(2r(r-1)+M+2)/(2(2M+3))}$
$\left\vert -\frac{(2r-1)\,e}{3}, \downarrow, \mathbf{3}^*\right\rangle$	$\left(\frac{M+2-r}{2M+3}, \frac{M+2-r}{2M+3}\right)$	(r, r)	$x_\downarrow (x_\uparrow x_\downarrow)^{-r}\, q^{(2r(r-1)+M+2)/(2(2M+3))}$

We claim that (D.6) equals the $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ character

$$Z_{\text{tot}} = \frac{1}{(q)_\infty^2} \sum_{p_i \in \mathbb{Z}} (x_\uparrow^{p_1} x_\downarrow^{p_2})\, q^{\frac{1}{2}\, \mathbf{p} \cdot \mathbf{K}_\phi \cdot \mathbf{p}} \tag{D.7}$$

corresponding to the partition function of two chiral bosons on the deformed weight lattice of $\mathfrak{sl}_3$. E.g., (D.7) for $M=0$ is precisely the Frenkel–Kac character (see, e.g., ref. 52) of the sum of the integrable highest weight modules of $(\widehat{\mathfrak{sl}_3})$ at level $k=1$.

To prove this claim, first observe that we can rewrite (D.6) as a sum over $2M+3$ sectors by using (5.8). Specifically,

$$Z_\infty\left(\mathbf{K}_\phi, \begin{pmatrix} \frac{2M+3-r}{2M+3} \\ \frac{2M+3-r}{2M+3} \end{pmatrix}\right) + x_\downarrow q^{(M+2-2r)/(2(2M+3))} Z_\infty\left(\mathbf{K}_\phi, \begin{pmatrix} \frac{M+2-r}{2M+3} \\ \frac{M+2-r}{2M+3} \end{pmatrix}\right)$$

$$= Z_\infty\left(\mathbf{K}_\phi, \begin{pmatrix} \frac{2M+3-r}{2M+3} \\ \frac{-r}{2M+3} \end{pmatrix}\right) \tag{D.8}$$

after which the claim follows by applying the statements of Theorem 4.1 and Corollary 5.2 in ref. 31.

Note that even though we use $3M+4$ sectors in generating the entire spectrum from the recursion relations (D.1) and (D.4), the $(\widehat{\mathfrak{sl}_3})_{k=1,M}$ partition function (D.7) can be written in terms of UCPFs based on $\mathbf{K} = \mathbf{K}_e \oplus \mathbf{K}_\phi$ using only $2M+3$ sectors. So, even though the UCPF form of a partition function is not unique, and we do not understand the precise relation between the number of sectors and the torus degeneracy in the sense of Wen *et al.*,[46] it is satisfying to see that the number $2M+3$ equals $\det \mathbf{K}_e$ which is the torus degeneracy for abelian fqH systems.

ACKNOWLEDGMENTS

We would like to thank Sathya Guruswamy, Nick Read and Ole Warnaar for useful discussions. This research was financially supported in part by the foundation FOM of the Netherlands and the Australian Research Council.

REFERENCES

1. W. Pan, J.-S. Xia, V. Shvarts, E. D. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Exact quantization of even-denominator fractional quantum Hall state at $\nu = 5/2$ Landau level filling factor, *Phys. Rev. Lett.* **83**:3530 (1999) [cond-mat/9907356].
2. G. Moore and N. Read, Nonabelions in the fractional quantum Hall effect, *Nucl. Phys. B* **360**:362 (1991).
3. A. M. M. Pruisken, B. Skoric, and M. A. Baranov, (Mis)handling gauge invariance in the theory of the quantum Hall effect III: the instanton vacuum and chiral edge physics, *Phys. Rev. B* **60**:16838 (1999) [cond-mat/9807241]; B. Skoric and A. M. M. Pruisken, The fractional quantum Hall effect: Chern–Simons mapping, duality, Luttinger liquids and the instanton vacuum, *Nucl. Phys. B* **559**[FS]:637 (1999) [cond-mat/9812437].
4. S. B. Isakov, G. S. Canright, and M. D. Johnson, Exclusion statistics for fractional quantum Hall states on a sphere, *Phys. Rev. B* **55**:6727 (1997) [cond-mat/9608139].
5. Y.-S. Wu and Y. Yu, Bosonization of one-dimensional exclusons and characterization of Luttinger liquids, *Phys. Rev. Lett.* **75**:890 (1995).
6. R. van Elburg and K. Schoutens, Quasi-particles in fractional quantum Hall effect edge theories, *Phys. Rev. B* **58**:15704 (1998) [cond-mat/9801272].
7. T. Fukui and N. Kawakami, Haldane's fractional exclusion statistics for multicomponent systems, *Phys. Rev. B* **51**:5239 (1995) [cond-mat/9408015].
8. F. D. M. Haldane, "Fractional statistics" in arbitrary dimensions: A generalization of the Pauli principle, *Phys. Rev. Lett.* **67**:937 (1991).
9. K. Schoutens, Exclusion statistics for non-abelian quantum Hall states, *Phys. Rev. Lett.* **81**:1929 (1998) [cond-mat/9803169].
10. E. Ardonne and K. Schoutens, New class of non-abelian spin-singlet quantum Hall states, *Phys. Rev. Lett.* **82**:5096 (1999) [cond-mat/9811352].
11. S. Guruswamy and K. Schoutens, Non-abelian exclusion statistics, *Nucl. Phys. B* **556**:530 (1999) [cond-mat/9903045].
12. P. Bouwknegt, L.-H. Chim, and D. Ridout, Exclusion statistics in conformal field theory and the UCPF for WZW models, *Nucl. Phys. B* **572**:547 (2000) [hep-th/9903176].
13. P. Bouwknegt and K. Schoutens, Exclusion statistics in conformal field theory—generalized fermions and spinons for level-1 WZW theories, *Nucl. Phys. B* **547**:501 (1999) [hep-th/9810113].
14. E. Ardonne, P. Bouwknegt, S. Guruswamy, and K. Schoutens, *K*-matrices for non-abelian quantum Hall states, *Phys. Rev. B* **61**:10298 (2000) [cond-mat/9908285].
15. X.-G. Wen, Topological orders and edge excitations in fractional quantum Hall states, *Adv. Phys.* **44**:405 (1995).
16. X.-G. Wen and A. Zee, Shift and spin vector: New topological quantum numbers for the Hall fluids, *Phys. Rev. Lett.* **69**:953 (1992); erratum, *Phys. Rev. Lett.* **69**:3000 (1992).
17. X.-G. Wen and A. Zee, Classification of abelian quantum Hall states and matrix formulation of topological fluids, *Phys. Rev. B* **46**:2290 (1992).
18. N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries, and the fractional quantum Hall effect, *Phys. Rev. B*, to appear [cond-mat/9906453].
19. S. B. Isakov, Generalization of statistics for several species of identical particles, *Mod. Phys. Lett. B* **8**:319 (1994); A. Dasnières de Veigy and S. Ouvry, Equation of state of an anyon gas in a strong magnetic field, *Phys. Rev. Lett.* **72**:600 (1994); Y.-S. Wu, Statistical distribution for generalized ideal gas of fractional-statistics particles, *Phys. Rev. Lett.* **73**:922 (1994).

20. K. Schoutens, Exclusion statistics in conformal field theory spectra, *Phys. Rev. Lett.* **79**:2608 (1997) [cond-mat/9706166].
21. M. Takahashi, One-dimensional Heisenberg model at finite temperature, *Prog. Theor. Phys.* **46**:401 (1971).
22. Al. B. Zamolodchikov, Thermodynamic Bethe Ansatz for RSOS scattering theories, *Nucl. Phys. B* **358**:497 (1991).
23. P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
24. N. Read and E. Rezayi, Beyond paired quantum Hall states: parafermions and incompressible states in the first excited Landau level, *Phys. Rev. B* **59**:8084 (1999) [cond-mat/9809384].
25. M. Greiter and F. Wilczek, Exact solutions and the adiabatic heuristic for quantum Hall states, *Nucl. Phys. B* **370**:577 (1992); M. Greiter, X.-G. Wen, and F. Wilczek, Paired Hall states, *Nucl. Phys. B* **374**:567 (1992).
26. E. Ardonne, N. Read, E. Rezayi, and K. Schoutens, paper in preparation.
27. J. Fröhlich, T. Kerler, U. M. Studer, and E. Thiran, Structuring the set of incompressible quantum Hall fluids, *Nucl. Phys. B* **453**:670 (1995) [hep-th/9505156].
28. A. Berkovich and B. McCoy, The universal chiral partition function for exclusion statistics, in *Statistical Physics on the Eve of the 21st Century*, Series on Adv. in Stat. Mech., Vol. 14, M. T. Batchelor and L. T. Wille, eds. (World Scientific, Singapore, 1999), pp. 240–256 [hep-th/9808013].
29. B. Richmond and G. Szekeres, Some formulas related to dilogarithms, the zeta function and the Andrews-Gordon identities, *J. Austral. Math. Soc.* (*Series A*) **31**:362 (1981); W. Nahm, A. Recknagel, and M. Terhoeven, Dilogarithm identities in Conformal Field Theory, *Mod. Phys. Lett. A* **8**:1835 (1993) [hep-th/9211034]; A. Kirillov, Dilogarithm identities, *Prog. Theor. Phys. Suppl.* **118**:61 (1995) [hep-th/9408113].
30. S. Dasmahapatra, R. Kedem, T. Klassen, B. McCoy, and E. Melzer, Quasi-Particles, Conformal Field Theory, and q-Series, *Int. J. Mod. Phys. B* **7**:3617 (1993) [hep-th/9303013].
31. P. Bouwknegt, Multipartitions, generalized Durfee squares and affine Lie algebra characters [math. CO/0002223].
32. F. D. M. Haldane, "Spinon gas" description of the $S=1/2$ Heisenberg chain with inverse-square exchange: exact spectrum and thermodynamics, *Phys. Rev. Lett.* **66**:1529 (1991).
33. D. Bernard, V. Pasquier, and D. Serban, Spinons in conformal field theory, *Nucl. Phys. B* **428**:612 (1994) [hep-th/9404050].
34. P. Bouwknegt, A. Ludwig, and K. Schoutens, Spinon bases, Yangian symmetry and fermionic representations of Virasoro characters in conformal field theory, *Phys. Lett. B* **338**:448 (1994) [hep-th/9406020].
35. P. Bouwknegt, A. Ludwig, and K. Schoutens, Spinon basis for higher level SU(2) WZW models, *Phys. Lett. B* **359**:304 (1995) [hep-th/9412108].
36. R. B. Laughlin, Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations, *Phys. Rev. Lett.* **50**:1395 (1983).
37. H. Frahm and M. Stahlsmeier, Spinon statistics in integrable spin-S Heisenberg chains, *Phys. Lett. A* **250**:293 (1998) [cond-mat/9803381].
38. B. L. Feigin and A. V. Stoyanovsky, Quasi-particles models for the representations of Lie algebra] and geometry of flag manifold, [hep-th/9308079]; Functional models for representations of current algebras and semi-infinite Schubert cells, *Funct. Anal. and Appl.* **28**:55 (1994).
39. G. Georgiev, Combinatorial constructions of modules for infinite-dimensional Lie algebras, I. Principal subspace, *J. Pure Appl. Algebra* **112**:247 (1996) [hep-th/9412054].
40. B. I. Halperin, Theory of the quantized Hall conductance, *Helv. Phys. Acta* **56**:75 (1983).

41. P. Bouwknegt and K. Schoutens, The $\widehat{SU(n)_1}$ WZW model]: Spinon decomposition and Yangian structure, *Nucl. Phys. B* **482**:345 (1996) [hep-th/9607064].
42. P. Bouwknegt and N. Halmagyi, q-identities and affinized projective varieties, II. Flag varieties, *Commun. Math. Phys.* **210**:663 (2000) [math-ph/9903033].
43. A. Berkovich, B. McCoy, and A. Schilling, Rogers–Schur–Ramanujan type identities for the $M(p, p')$ minimal models of conformal field theory, *Commun. Math. Phys.* **191**:325 (1998) [q-alg/9607020].
44. A. Kirillov, Dilogarithm identities, *Prog. Theor. Phys. Suppl.* **118**:61 (1995) [hep-th/9408113].
45. G. Hatayama, A. Kirillov, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, Character formulae of $\widehat{\mathfrak{sl}_n}$-modules and inhomogeneous paths, *Nucl. Phys. B* **536**:575 (1999) [math.QA/9802085].
46. X.-G. Wen, Vacuum degeneracy of chiral spin states in compactified space, *Phys. Rev. B* **40**:7387 (1989); ibid., Topological orders in rigid states, *Int. J. Mod. Phys. B* **4**:239 (1990); ibid., Edge excitations in the fractional quantum Hall states at general filling fractions, *Mod. Phys. Lett. B* **5**:39 (1991); X.-G. Wen and Q. Niu, Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on higher genus Riemann surfaces, *Phys. Rev. B* **41**:9377 (1990); X.-G. Wen and A. Zee, Topological degeneracy of quantum Hall fluids [cond-mat/9711223].
47. E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, A Chern–Simons effective field theory for the Pfaffian quantum Hall state, *Nucl. Phys. B* **516**:704 (1998) [cond-mat/9711087]; E. Fradkin, C. Nayak, and K. Schoutens, Landau-Ginzburg theories for non-abelian quantum Hall states, *Nucl. Phys. B* **546**:711 (1999) [cond-mat/9811005]; D. C. Cabra, E. Fradkin, G. L. Rossini, and F. A. Schaposnik, Non-Abelian fractional quantum Hall states and chiral coset conformal field theories [cond-mat/9905192].
48. N. Read and E. Rezayi, Quasiholes and fermionic zero modes of paired fractional quantum Hall states: The mechanism for non-Abelian statistics, *Phys. Rev. B* **54**:16864 (1996) [cond-mat/9609079].
49. V. Gurarie and E. Rezayi, Parafermion statistics and quasihole excitations for the generalizations of the paired quantum Hall states, *Phys. Rev. B* **61**:5473 (2000) [cond-mat/9812288].
50. G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge University Press, Cambridge, 1990).
51. J. L. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, 1966).
52. V. G. Kac, *Infinite Dimensional Lie Algebras* (Cambridge University Press, Cambridge, 1985).